The 2016 Nobel prize in Physics

D. Thouless and Topological Invariants

J. Avron

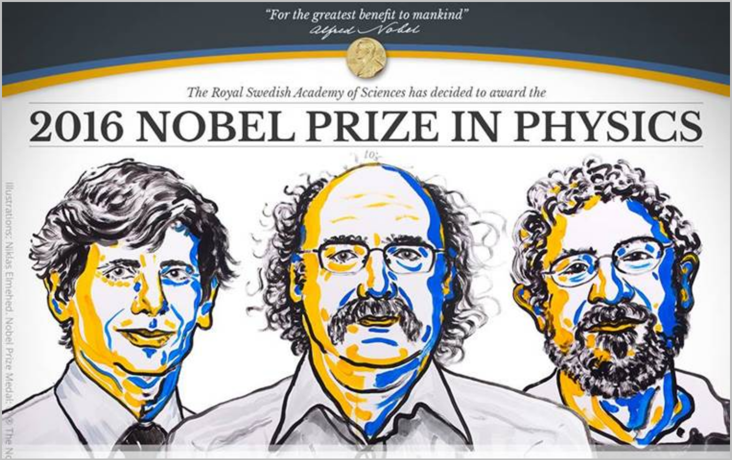
May 2017

There is geometry in the humming of the strings, there is music in the spacing of the spheres. (Pythagoras)

Happy birthday, Petr

D. Thouless, D. Haldane, M. Kosteritz

Kosterlitz-Thouless transition; TKNN aka Chern numbers



Mathematical physics in 2 D

Kosterlitz-Thouless transition; TKNN aka Chern numbers

Quantum transistors

Marginal phase transition

TKNN 1982

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Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[a] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

The Hall conductance of a two-dimensional electron gas has been studied in a uniform magnetic field and a periodic substrate potential U. The Kubo formula is written in a form that makes apparent the quantization when the Fermi energy lies in a gap. Explicit expressions have been obtained for the Hall conductance for both large and small $U/\hbar\omega_c$.

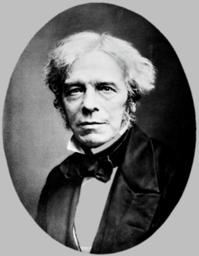
PACS numbers: 72.15.Gd, 72.20. Mg, 73.90.+b

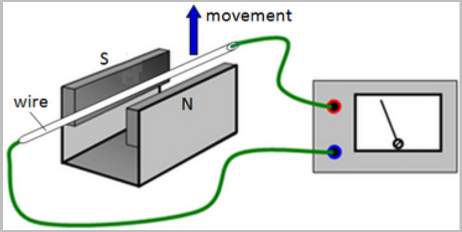
The experimental discovery by von Klitzing, Dorda, and Pepper[1] of the quantization of the Hall conductance of a two-dimensional electron gas in

Laughlin's[2] argument that the Hall conductance is quantized whenever the Fermi energy lies in an energy gap, even if the gap lies within a Landau

TKNN: Topological quantum numbers (1982)
B. Simon: Chern classes in QM (1983)
M.V. Berry: Adiabatic curvature, Berry's phase (1984)

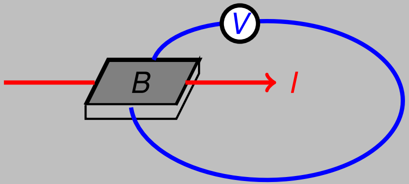
Maxwell’s Ingenious blunders

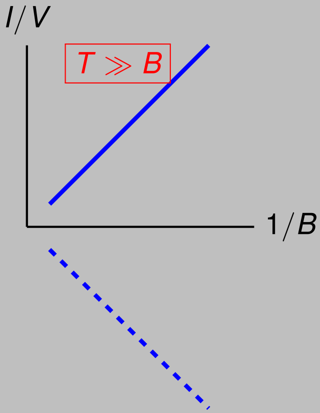




The Classical Hall effect

1879



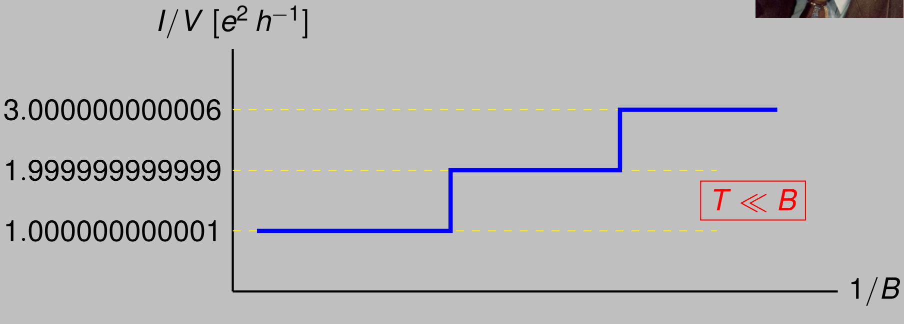


The Quantum Hall effect

von Klitzing (Nobel 1985)

- Quantum unit of resistance

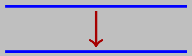
$$\frac{h}{e^2} \approx 26\ [K\Omega]$$



Fundamental vs natural standards

Time: Natural but not fundamental

- Second: Hyperfine transition of Cs_{133}



$9,192,631,770\ [Hz]$

All Cs_{133} atoms are equal

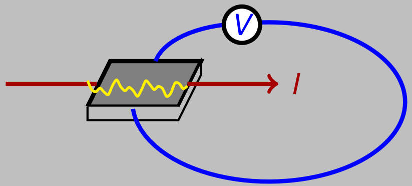
Natural, NOT fundamental

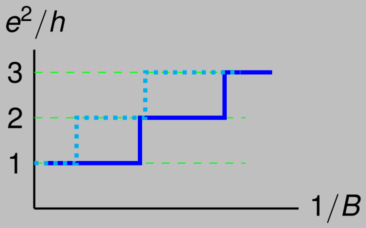
$9,192,631,770\ [Hz]$ is precisely measurable, but
Not related to a fundamental time scale in a known way

Fundamental vs natural standards

Ohm: Artificial but fundamental

- Ohm: QHE





Every transistor is different

Artificial but fundamental

Resistance:

The quantum unit of resistance is precisely measurable

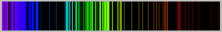
Heisenberg quantization

Quantization and spectral theory

- Observables=Linear operators
- Measurements yield eigenvalues

$$\mathrm{Spect}(L_z) \subseteq \frac{\hbar}{2}\,\mathbb{Z}$$

- Not the mechanism in the QHE



Dirac quantization

Electric charges are an integer

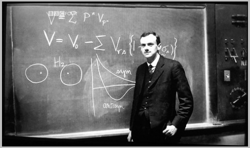
- Electric charges $\in q_e \mathbb{Z}$

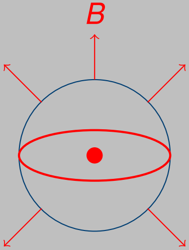
$$q_{electron} = -q_{proton}$$

- Magnetic monopole q_{mag}

$$q_{electric} q_{mag} \in \frac{\hbar c}{2} \mathbb{Z}$$

- Not the mechanism in the QHE



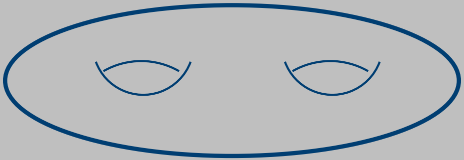


TKNN quantization

Quantization of transport coefficients

Topological invariants & Transport

Hall Conductance= Chern number



Gauss-Bonnet-Chern

$$\frac{1}{2\pi}\int Curvature \in \mathbb{Z}$$

Real scientists solve models. Wimps generalize M. Berry

The Hofstadter model

- $\psi(n,m) \in \ell^2(\mathbb{Z}^2)$
- North translation

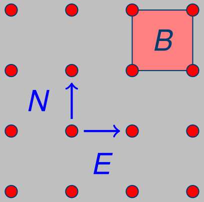
$$(N\psi)(n,m) = \psi(n,m-1)$$

- East translations:

$$(E\psi)(n,m) = e^{-2\pi i B\, m}\psi(n-1,m)$$

- Hamiltonian

$$H = E + N + h.c.$$



Periodic matrices

The importance of families

- When $B = \frac{p}{q}$: H is periodic and reduces to

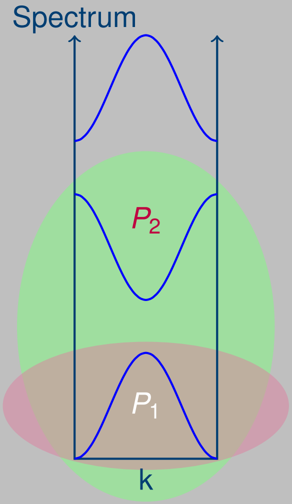
$$\underbrace{H(k_1, k_2)}_{q \times q \text{ periodic matrix}} = e^{ik_1} \underbrace{\mathbf{T}}_{\text{cyclic shift}} + e^{ik_2} \underbrace{\hat{\mathbf{T}}}_{\text{FT of cyclic shift}} + h.c.$$

- Example $B = \frac{1}{3}$:

$$\mathbf{T} = \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}}_{3 \times 3}, \quad \hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad \underbrace{\omega = e^{2\pi i B}}_{\text{root of unity}}$$

Quantum states as bundles of projections

Full bands of free Fermions

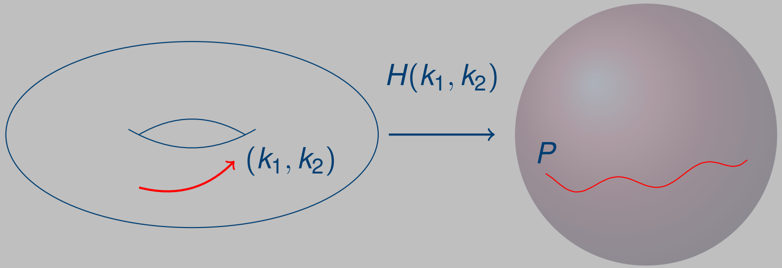


- $H(k_1, k_2)$: periodic $q \times q$ hermitian matrix
- $Spectrum(H)$: q-bands.
- $P_j(k_1, k_2)$: bundles of projections

Quantum states at $T = 0$

- Finite system: Rank 1 projection
- Full bands of free Fermions: Bundle of projections

Families of spectral projections



- Smooth, periodic spectral projection

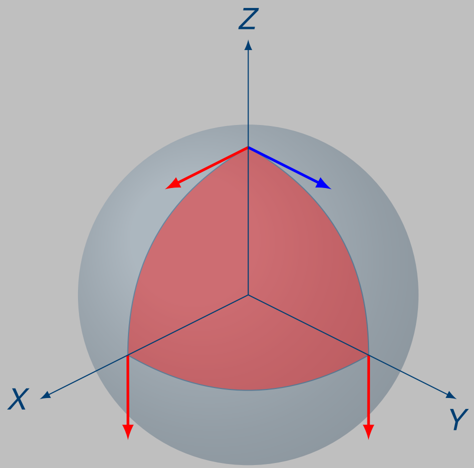
$$P(k_1, k_2) = P(k_1 + 2\pi, k_2) = P(k_1, k_2 + 2\pi)$$

- Rank one projection:

$$\underbrace{P = |\psi\rangle\langle\psi|}_{\textit{family}}$$

Curvature in differential geometry

Curvature: Failure of parallel transport



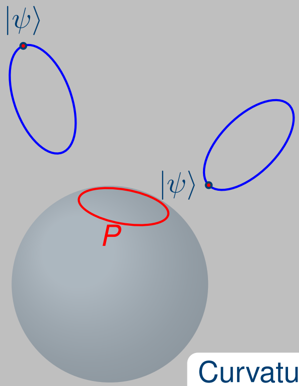
Curvature

$$\frac{\text{Failure of parallel transport}}{\textit{area}}$$

$$\frac{\pi/2}{4\pi R^2/8} = \frac{1}{R^2}$$

Curvature of bundles of projections in Hilbert space

Berry's phase: Failure of parallel transport



- Berry's (gauge) 1-form

$$\mathbf{A} = i \langle \psi | \, \boldsymbol{d}_k \psi \rangle$$

- Berry's phase: Failure of parallel transport

$$\int_{\square} \mathbf{A} = \int_{\blacksquare} \mathbf{dA}$$

Curvature: Local failure of parallel transport

$$\underbrace{\boldsymbol{dA}}_{2-form} \Longleftrightarrow (dA)_{jk}(\phi) = -2\,\mathrm{Im}\,\langle \partial_j \psi | \, \partial_k \psi \rangle$$

Expectation of currents=Rates of Berry's gauge

- Evolution equation

$$i\frac{d\,|\psi_t\rangle}{dt} = H(k,t)\,|\psi_t\rangle$$

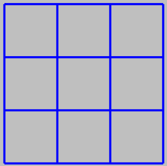
- Define current:

$$\frac{\partial H}{\partial k}$$

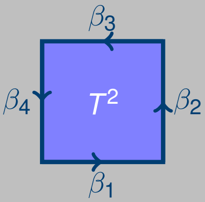
- Expectations related to rate of Berry's phase

$$\underbrace{\left\langle \psi_t \left| \frac{\partial H}{\partial k} \right| \psi_t \right\rangle}_{\text{expectation of current}} = \underbrace{i\frac{d}{dt}\,\langle \psi | \partial_k \psi\rangle}_{\text{rate of Berry's phase}}$$

Hall conductance=Chern number



$T^2 = \mathbb{R}^2/\mathbb{Z}^2$



TKNN

$$\text{Hall conductance} = \frac{1}{2\pi}\int_{T^2} \boldsymbol{dA}$$

Gauss-Bonnet-Chern rediscovered

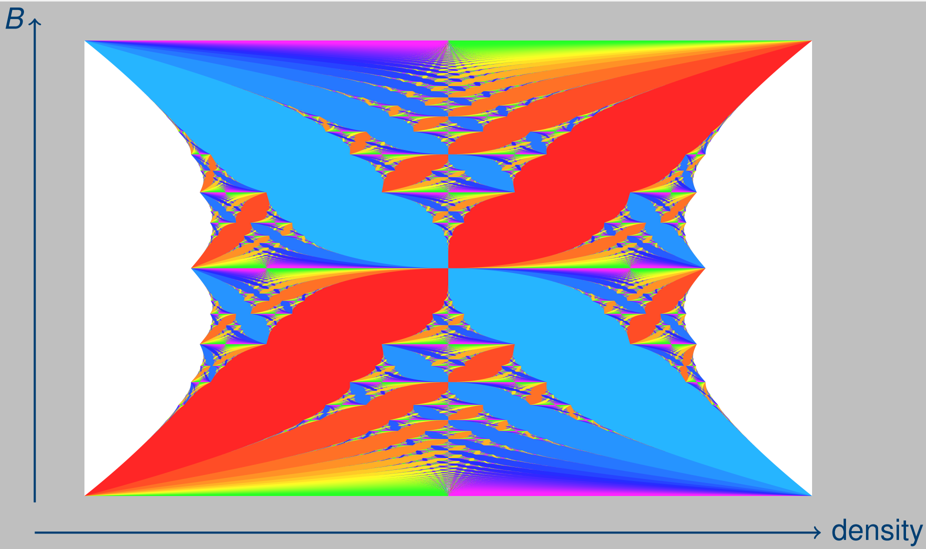
$$\frac{1}{2\pi}\int_{T^2} \boldsymbol{dA} \in \mathbb{Z}$$

$$\int_{T^2} \mathbf{dA} = \oint_{\partial T^2} \mathbf{A}$$

$$= \beta_1 + \beta_2 + \beta_3 + \beta_4 \in 2\pi\mathbb{Z}$$

Hofstadter butterfly

Fractal diagram of Chern numbers



What have we learned?

And what did I not cover

- Transport coefficients have geometric significance
- Macroscopic systems of Fermions are bundles of projections
- Bundles of projections are related to Chern classes
- K-theory, Entanglement, Topological states of matter,....

2016 Nobel prize food for thought

Fiber bundles with cream cheese