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Semiclassical bounds for magnetic Laplacian

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UNIVERSITAS OSTRAVIENSIS

joint work with Pavel Exner, Timo Weidl and Hynek Kovarik

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. We denote:

$-\Delta_D^\Omega$ – the Dirichlet Laplacian in $L_2(\Omega)$

The spectrum of the operator $-\Delta_D^\Omega$ is purely discrete. We denote by

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots$$

the ordered sequence of its eigenvalues (counting multiplicities).

In the present talk we consider a magnetic version of the Dirichlet Laplacian, the operator

$$H_\Omega(A) = (i\nabla + A(x))^2$$

associated with the closed form

$$\| (i\nabla + A)u \|^2_{L^2(\Omega)}, \quad u \in \mathcal{H}^1_0(\Omega),$$

with the real valued and sufficiently smooth vector function A.

The field A is called the magnetic potential.

The magnetic Sobolev norm

$$\| (i\nabla + A)u \|^2_{L^2(\Omega)}, \quad u \in \mathcal{H}^1_0(\Omega),$$

is equivalent to the non magnetic one (Ω is a bounded domain), whence

$H_\Omega(A)$ has a purely discrete spectrum

We shall denote the eigenvalues of $H_\Omega(A)$ by

$$\lambda_k = \lambda_k(\Omega, A)$$

assuming that they repeat according to their multiplicities.

The object of our interest in this talk are bounds of the eigenvalue moments of such operators.

At first we mention some known results for the Dirichlet Laplacian.

In what follows by $(\cdot)_+$ we denote the positive part of the quantity staying in the brackets, namely

$$(f)_+ = \begin{cases} f, & f \geq 0, \\ 0, & f < 0. \end{cases}$$

Weyl asymptotics

$$\sum_k (\Lambda - \lambda_k(\Omega))_+^\sigma = L^{\mathrm{cl}}_{\sigma,d}|\Omega|\Lambda^{\sigma+\frac{d}{2}} + o(\Lambda^{\sigma+\frac{d}{2}}), \quad \sigma \geq 0, \quad \Lambda \to \infty,$$

where $|\Omega|$ is the volume of Ω and

$$L^{\mathrm{cl}}_{\sigma,d} = \frac{\Gamma(\sigma+1)}{(4\pi)^{\frac{d}{2}}\Gamma(\sigma+1+d/2)}. \qquad (\star)$$

Berezin bound

$$\sum_k (\Lambda - \lambda_k(\Omega))_+^\sigma \leq L^{\mathrm{cl}}_{\sigma,d}\,|\Omega|\,\Lambda^{\sigma+\frac{d}{2}} \quad \text{for any } \sigma \geq 1 \text{ and } \Lambda > 0,$$

where $L^{\mathrm{cl}}_{\sigma,d}$ is defined by $(\star)$.

The constant $L^{\mathrm{cl}}_{\sigma,d}$ is optimal.

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The constant $L_{\sigma,d}^{\mathrm{cl}}$ is optimal.

Remark 1

A. Laptev proved that Berezin bound

$$\sum_k (\Lambda - \lambda_k(\Omega))_+^\sigma \leq L_{\sigma,d}^{\mathrm{cl}} \, |\Omega| \, \Lambda^{\sigma+\frac{d}{2}}, \quad \Lambda > 0$$

holds true for $0 \leq \sigma < 1$ as well, but with another, probably non-sharp constant on the right-hand side.

Namely, one has for $0 \leq \sigma < 1$

$$\sum_k (\Lambda - \lambda_k(\Omega))_+^\sigma \leq 2 \left(\frac{\sigma}{\sigma+1} \right)^\sigma L_{\sigma,d}^{\mathrm{cl}} \, |\Omega| \, \Lambda^{\sigma+\frac{d}{2}}, \quad \Lambda > 0.$$

Remark 2

In particular case $\sigma = 1$ Berezin bound has the form

$$\sum_k (\Lambda - \lambda_k(\Omega))_+ \leq L^{\mathrm{cl}}_{1,d} \, |\Omega| \, \Lambda^{1+\frac{d}{2}}, \quad \Lambda > 0,$$

and is equivalent to the lower bound

$$\sum_{j=1}^{N} \lambda_j(\Omega) \geq C_d |\Omega|^{-\frac{2}{d}} N^{1+\frac{2}{d}}, \quad C_d = \frac{4\pi d}{d+2} \Gamma(d/2+1)^{\frac{2}{d}}.$$

This estimate is called Li–Yau inequality.

$$H_\Omega(A) = (i\nabla + A(x))^2 \quad \text{– magnetic Laplacian}$$

In view of the pointwise diamagnetic inequality

$$|\nabla|u(x)|| \leq |(i\nabla + A)u(x)| \quad \text{for a.a. } x \in \Omega$$

one has

$$\lambda_1(\Omega, A) \geq \lambda_1(\Omega, 0).$$

However for $j \geq 2$ the estimate $\lambda_j(\Omega, A) \geq \lambda_j(\Omega, 0)$ fails in general.

Nevertheless, momentum estimates are still valid for some values of the parameters.

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Laptev–Weidl, 2000

The sharp bound

$$\sum_k (\Lambda - \lambda_k(\Omega, A))_+^\sigma \le L_{\sigma,d}^{\mathrm{cl}} \, |\Omega| \, \Lambda^{\sigma+\frac{d}{2}} \quad \text{for any } \; \Lambda > 0$$

holds true for arbitrary magnetic fields provided $\sigma \ge \frac{3}{2}$, and the same sharp bound holds true for constant magnetic fields if $\sigma \ge 1$.

Frank–Loss–Weidl, 2009

In the dimension $d = 2$ the bound

$$\sum_k (\Lambda - \lambda_k(\Omega, A))_+^\sigma \le 2 \left(\frac{\sigma}{\sigma + 1} \right)^\sigma L_{\sigma,d}^{\mathrm{cl}} \, |\Omega| \, \Lambda^{\sigma+\frac{d}{2}}$$

holds true for constant magnetic fields if $0 \le \sigma < 1$, moreover the constant on its right-hand side cannot be improved.

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The sharp bound

$$\sum_k (\Lambda - \lambda_k(\Omega, A))_+^\sigma \leq L^{\mathrm{cl}}_{\sigma,d}\, |\Omega|\, \Lambda^{\sigma+\frac{d}{2}} \quad \text{for any } \Lambda > 0$$

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holds true for constant magnetic fields if $0 \leq \sigma < 1$, moreover the constant on its right-hand side cannot be improved.

Our first aim in this work is to derive a two-dimensional version of the Li–Yau inequality in presence of a constant magnetic field giving rise to an extra term on the right-hand side.

Note, that in two-dimensional case Li–Yau inequality has the form

$$\sum_{j=1}^{N} \lambda_j(\Omega, 0) \geq \frac{2\pi N^2}{|\Omega|}.$$

Suppose that the motion is confined to a domain $\omega \subset \mathbb{R}^2$ being exposed to influence of a constant magnetic field of intensity B, and let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector potential generating this field.

We denote by

$$H_\omega(A) = (i\nabla + A(x))^2$$

the corresponding magnetic Dirichlet Laplacian on ω and by

$$0 < \lambda_1(\omega, A) \leq \lambda_2(\omega, A) \leq \lambda_3(\omega, A) \leq \dots$$

the sequence of its eigenvalues arranged in the ascending order with account of their multiplicity.

Theorem [Weidl–Kovarik, 2013]

Let $\omega \subset \mathbb{R}^2$ be bounded and convex. Then for any $N \in \mathbb{N}$ it holds

$$\sum_{j=1}^{N} \lambda_j(\omega, A) \geq \frac{2\pi N^2}{|\omega|} + \frac{1}{64} \frac{\sigma^2(\omega)}{|\omega|^2},$$

where ...

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Notations

- $\delta(x) := \min_{y\in\partial\omega} |x-y|$ – the distance from x to the boundary of ω
- for $\beta > 0$ we set $\omega_\beta := \{x \in \omega : \ \delta(x) < \beta\}$,
- $$\sigma(\omega) := \inf_{0<\beta<R(\omega)} \frac{|\omega_\beta|}{\beta},$$

  where $R(\omega) := \sup_{x\in\omega} \delta(x)$ is the in-radius of ω.

Our first aim is to extend the inequality

$$\sum_{j=1}^{N} \lambda_j(\omega, A) \geq \frac{2\pi N^2}{|\omega|} + \frac{1}{64} \frac{\sigma^2(\omega)}{|\omega|^2}.$$

obtained by Weidl and Kovarik, but with an additional term on the right-hand side depending on B only and independent of the geometry of ω.

Theorem 1 [B–Exner–Kovarik–Weidl, 2016]

Assume that $\omega \subset \mathbb{R}^2$ is a bounded domain. Then the inequality

$$\sum_{j=1}^{N} \lambda_j(\omega, A) \geq \frac{2\pi N^2}{|\omega|} + \frac{B^2}{2\pi}|\omega| m(1-m)$$

holds, where $m := \left\{ \frac{2\pi N}{B|\omega|} \right\}$ is the fractional part of $\frac{2\pi N}{B|\omega|}$.

Since $0 \leq m < 1$ by definition the last term can regarded as a non-negative remainder term.

Theorem 2 [B–Exner–Kovarik–Weidl, 2016]

Let $\omega \subset \mathbb{R}^2$ be a bounded domain, and $\Lambda > B$. Then

$$\sum_{j=1}^{N} (\Lambda - \lambda_j(\omega, A))_+ \leq \frac{(\Lambda^2 - B^2)|\omega|}{8\pi} + \frac{(\Lambda - B)B|\omega|}{4\pi} \left\{ \frac{\Lambda + B}{2B} \right\}.$$

Spectral analysis simplifies if the domain ω allows for a separation of variables.

If the magnetic field is non-constant but still radially symmetric, in general one cannot find the eigenvalues explicitly but it possible to find a bound to the eigenvalue moments in terms of an appropriate radial two-dimensional Schrödinger operator.

Theorem 4 [B–Exner–Kovarik–Weidl, 2016]

Let $H_\omega(A)$ be the magnetic Dirichlet Laplacian $H_\omega(A)$ on a disc ω of radius $r_0 > 0$ centered at the origin with a radial magnetic field $B(x) = B(|x|)$. Assume that

$$\alpha := \int_0^{r_0} sB(s)\,\mathrm{d}s < \frac{1}{2}\,.$$

Then for any Λ, $\sigma \geq 0$, the following inequality holds true

$$\mathrm{tr}(\Lambda - H_\omega(A))_+^\sigma \leq \left(\frac{1}{\sqrt{1-2\alpha}} + \sup_{n\in\mathbb{N}} \left\{ \frac{n}{\sqrt{1-2\alpha}} \right\} \right)$$
$$\times\,\mathrm{tr}\left(\Lambda - \left(-\Delta_D^\omega + \frac{1}{x^2+y^2}\left(\int_0^{\sqrt{x^2+y^2}} sB(s)\,\mathrm{d}s\right)^2\right)\right)_+^\sigma .$$

Remark

$$\inf \sigma(H_\omega(A)) \geq \inf \sigma \left(-\Delta_D^\omega + \frac{1}{x^2+y^2} \left(\int_0^{\sqrt{x^2+y^2}} sB(s)\,\mathrm{d}s \right)^2 \right).$$

Let $-\Delta_\Omega$ be the Dirichlet Laplacian on a domain $\Omega \subset \mathbb{R}^3$.

Theorem [Laptev–Weidl, 2000]

For any $\sigma \geq \frac{3}{2}$ one has the inequality

$$\operatorname{tr}\left(\Lambda - (-\Delta_\Omega)\right)_+^\sigma \leq L_{1,\sigma}^{\mathrm{cl}} \int_{\mathbb{R}} \operatorname{tr}\left(\Lambda - (-\Delta_{\omega(x_3)})\right)_+^{\sigma+\frac{1}{2}} \, \mathrm{d}x_3,$$

where $-\Delta_{\omega(x_3)}$ is the Dirichlet Laplacian on the section

$$\omega(x_3) = \left\{x' = (x_1, x_2) \in \mathbb{R}^2 \mid x = (x', x_3) = (x_1, x_2, x_3) \in \Omega\right\},$$

Remark: The integral at the right-hand side in fact restricted to those x_3 for which $\inf \operatorname{spec}(-\Delta_{\omega(x_3)}) < \Lambda$.

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A similar technique can be used also in the magnetic case.

$$A = (A_1, A_2, A_3) : \Omega \to \mathbb{R}^3$$
$$\mathcal{H}_\Omega(A) = (i\nabla - A(x))^2 \quad \text{on} \quad L^2(\Omega)$$

For the fixed x_3

$$\widetilde{A}(x) := (A_1(x), A_2(x)).$$
$$\widetilde{H}_{\omega(x_3)}(\widetilde{A}) = (i\nabla_{(x_1,x_2)} - \widetilde{A}(x))^2 \quad \text{on} \quad L^2(\omega(x_3)),$$

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Theorem [Laptev–Weidl, 2000]

For $\sigma \geq \frac{3}{2}$

$$\mathrm{tr}(\Lambda - \mathcal{H}_\Omega(A))_+^\sigma \leq L_{1,\sigma}^{\mathrm{cl}} \int_{\mathbb{R}} \mathrm{tr}(\Lambda - \widetilde{H}_{\omega(x_3)}(\widetilde{A}))_+^{\sigma+1/2} \, \mathrm{d}x_3$$

Note that for the **fixed** x_3 the two-dimensional vector potential $\widetilde{A}(x_1, x_2, x_3)$ corresponds to the magnetic field

$$\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} = B_3(x).$$

The class of fields to consider here are those of the form

$$B(x) = (B_1(x), B_2(x), B_3(x_3)).$$

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Theorem [B.-Exner-Kovarik-Weidl 2016]

$$\mathrm{tr}(\Lambda - \mathcal{H}_\Omega(A))_+^\sigma \le \frac{\Gamma(\sigma+3/2)\Lambda^{\sigma-1/2}}{4\pi(2\sigma-1)\Gamma(\sigma-1/2)}\, L_{1,\sigma}^{\mathrm{cl}} \int_{\{x_3:\, B_3(x_3)<\Lambda\}} |\omega(x_3)|$$

$$\times \left[\left(\Lambda^2 - B_3(x_3)^2\right) + 2B_3\left(\Lambda - B_3(x_3)\right) \left\{ \frac{\Lambda + B_3}{2B_3} \right\} \right] \mathrm{d}x_3$$

for any $\sigma \ge 3/2$.

Example (radial magnetic field)

Consider the same cusp-shaped region Ω in the more general situation when the third field component can depend on the radial variable, $B(x) = (B_1(x), B_2(x), B_3(x_1^2 + x_2^2, x_3))$, assuming that

$$\sup_{x_3 \in \mathbb{R}} \alpha(x_3) = \sup_{x_3 \in \mathbb{R}} \int_0^{r_0(x_3)} s B_3(s, x_3)\, \mathrm{d}s < \frac{1}{2}\,.$$

Then...

...

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$$\mathrm{tr}(\Lambda - \mathcal{H}_\Omega(A))_+^\sigma \leq L_{1,\sigma}^{\mathrm{cl}} \int_{\mathbb{R}} \left(\frac{1}{\sqrt{1-2\alpha(x_3)}} + \sup_{n\in\mathbb{N}} \left\{ \frac{n}{\sqrt{1-2\alpha(x_3)}} \right\} \right)$$

$$\times \mathrm{tr} \left(\Lambda - \left(-\Delta_D^{\omega(x_3)} + \frac{1}{x_1^2+x_2^2} \left(\int_0^{\sqrt{x_1^2+x_2^2}} sB_3(s,x_3)\,\mathrm{d}s \right)^2 \right) \right)_+^{\sigma+1/2}$$

for any $\sigma \geq 3/2$.

Now we change the topic and consider situations when the discrete spectrum comes from the magnetic field alone. We are going to demonstrate a Berezin-type estimate for the magnetic Laplacian on $\mathbb{R}^2$ with the field which is a radial and local perturbation of a homogeneous one.

$$H(B) = -\partial_x^2 + (i\partial_y + A_2)^2, \quad A = \left(0, B_0\, x - f(x,y)\right), \quad \text{on} \quad L^2(\mathbb{R}^2)$$

$$f(x,y) = -\int_x^\infty g(\sqrt{t^2+y^2})\, \mathrm{d}t\,.$$

with $g : \mathbb{R}_+ \to \mathbb{R}_+$.

The operator $H(B)$ is then associated with the magnetic field

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$$B = B(x,y) = B_0 - g(\sqrt{x^2+y^2})\,.$$

In the following we will suppose that

(i) the function $g \in L^\infty(\mathbb{R}_+)$ is non-negative and such that both f and $\partial_{x_2} f$ belong to $L^\infty(\mathbb{R}^2)$, and

$$\lim_{x_1^2+x_2^2\to\infty} \left(|\partial_{x_2} f(x_1, x_2)| + |f(x_1, x_2)|\right) = 0\,.$$

(ii) $\|g\|_\infty \le B_0\,.$

Let us turn back to the unperturbed case

$$H(B_0) = -\partial_x^2 + (i\partial_y + B_0 x)^2.$$

Then the corresponding spectrum consists of identically spaced eigenvalues of infinite multiplicity,

$$\sigma(H(B_0)) = \{(2n-1)B_0, \quad n \in \mathbb{N}\} .$$

$$A_0 = (0, a_0(r))\,, \qquad A = (0, a(r))\,,$$

with

$$a_0(r) = \frac{B_0 r}{2}\,, \qquad a(r) = \frac{B_0 r}{2} - \frac{1}{r}\int_0^r g(s)\, s\, \mathrm{d}s\,.$$

Finally let us denote by

$$\alpha = \int_0^\infty g(r)\, r\, \mathrm{d}r.$$

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Theorem 4 [B–Exner–Kovarik–Weidl, 2016]

Let the assumptions (i) and (ii) be satisfied, and suppose moreover that $\alpha \leq 1$. Then

$$\inf \sigma_{\mathrm{ess}}(H(B)) = B_0.$$

Theorem 4 [B–Exner–Kovarik–Weidl, 2016]

The inequality holds true

$$\mathrm{tr}(H(B) - B_0)_-^\gamma \le 2^\gamma \sum_{k=0}^{\infty} \Lambda_k^\gamma\,, \qquad \gamma \ge 0\,,$$

For every $k \in \mathbb{N}_0$

$$V_k(r) := \frac{2k}{r}(a_0(r) - a(r)) + a^2(r) - a_0^2(r)\,,$$

$$\psi_k(r) = \sqrt{\frac{B_0}{\Gamma(k+1)}}\,\left(\frac{B_0}{2}\right)^{k/2} r^k \exp\left(-\frac{B_0\, r^2}{4}\right).$$

$$\Lambda_k = \Big(\psi_k, \big(V_k(\cdot)\big)_- \psi_k\Big)_{L^2(\mathbb{R}_+, r\mathrm{d}r)}\,.$$

Let us return to the three-dimensional situation and consider a magnetic Hamiltonian $\mathcal{H}(B)$ in $L^2(\mathbb{R}^3)$ associated to the magnetic field $B : \mathbb{R}^3 \to \mathbb{R}^3$ regarded as a perturbation of a homogeneous magnetic field of intensity $B_0 > 0$ pointing in the x_3-direction,

$$B(x_1, x_2, x_3) = (0, 0, B_0) - b(x_1, x_2, x_3)\,,$$

with the perturbation b of the form

$$b(x_1, x_2, x_3) = \left(-\omega'(x_3)\, f(x_1, x_2),\, 0,\, \omega(x_3)\, g\left(\sqrt{x_1^2 + x_2^2}\right)\right).$$

Here $\omega : \mathbb{R} \to \mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ and

$$f(x_1, x_2) = -\int_{x_1}^{\infty} g\left(\sqrt{t^2 + x_2^2}\right) \mathrm{d}t\,.$$

The first component of B then ensures that $\nabla \cdot B = 0$, which is required by the Maxwell equations which include no magnetic monopoles; it vanishes if the field is x_3-independent.

A vector potential generating this field can be chosen in the form

$$A(x_1, x_2, x_3) = (0,\ B_0\, x_1 - \omega(x_3)\, f(x_1, x_2),\ 0)\,,$$

and

$$\mathcal{H}(B) = -\partial_{x_1}^2 + (i\partial_{x_2} + B_0\, x_1 - \omega(x_3)\, f(x_1, x_2))^2 - \partial_{x_3}^2.$$

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In the following we suppose that

(i) the function $g \in L^\infty(\mathbb{R}_+)$ is non-negative, such that f and $\partial_{x_2} f$ belong to $L^\infty(\mathbb{R}^2)$, and

$$\lim_{x_1^2+x_2^2\to\infty} \left(|\partial_{x_2} f(x_1,x_2)| + |f(x_1,x_2)|\right) = 0\,,$$

(ii) $\omega \geq 0$, $\omega \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and

$$\|\omega\|_\infty \, \|g\|_\infty \leq B_0\,, \qquad \lim_{|x_3|\to\infty} \omega(x_3) = 0\,.$$

Theorem 4 [B–Exner–Kovarik–Weidl, 2016]

The assumptions (i) and (ii) imply $\sigma_{\text{ess}}(\mathcal{H}(B)) = [B_0, \infty)$.

Now we are going to formulate Lieb-Thirring-type inequalities for the negative eigenvalues of $\mathcal{H}(B) - B_0$. We denote by

$$\alpha(x_3) = \omega(x_3) \int_0^\infty g(r)\, r\, \mathrm{d}r.$$

Theorem 4 [B–Exner–Kovarik–Weidl, 2016]

Let assumptions (i) and (ii) be satisfied. Suppose, moreover, that $\sup_{x_3} \alpha(x_3) \leq 1$ and put

$$\Lambda_k(x_3) = \left(\psi_k, \left(V_k(\cdot; x_3)\right)_- \psi_k\right)_{L^2(\mathbb{R}_+, r\mathrm{d}r)} .$$

Then the inequality holds true

$$\mathrm{tr}\,(\mathcal{H}(B) - B_0)_-^\sigma \leq L_{\sigma,1}^{\mathrm{cl}}\, 2^{\sigma+\frac{1}{2}} \int_{\mathbb{R}} \sum_{k=0}^\infty \Lambda_k(x_3)^{\sigma+\frac{1}{2}}\, \mathrm{d}x_3\,, \quad \sigma \geq \frac{3}{2}.$$

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Let D be a circle of radius R centered at the origin and put

$$g(r) = \begin{cases} B_0 & r \leq R \\ 0 & r > R \end{cases} .$$

Theorem 4 [B–Exner–Kovarik–Weidl, 2016]

Assume that $B_0 R^2 \leq 2$. Suppose moreover that $\|\omega\|_\infty \leq 1$. Then for any $\sigma > 3/2$ it holds

$$\operatorname{tr}\left(\mathcal{H}(B) - B_0\right)_-^\sigma \;\leq\; L_{\sigma,1}^{\mathrm{cl}}\, J\Big(B_0\,,\sigma\Big)\; B_0^{\sigma+\frac{1}{2}} \int_{\mathbb{R}} \omega(x_3)^{\sigma+\frac{1}{2}}\, \mathrm{d}x_3\,,$$

where

$$J(B_0,\sigma) = \left(B_0\, R^2\right)^{\sigma+\frac{1}{2}} \left(1 + \sum_{k=1}^{\infty} \left(\left(\frac{B_0\, R^2}{2}\right)^{k+1} \frac{1}{k!} + \frac{1}{2\sqrt{2\pi k}}\right)^{\sigma+\frac{1}{2}}\right) .$$

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Thank you for your attention!

Let us give a **sketch of the proof of the Berezin-type inequality**:

$$\sum_{j=1}^{N}(\Lambda - \lambda_j(A))_+ \leq \frac{(\Lambda^2 - B^2)|\omega|}{8\pi} + \frac{(\Lambda - B)B|\omega|}{4\pi}\left\{\frac{\Lambda + B}{2B}\right\}.$$

One has (Theorem 1) the Li-Yau-type inequality

$$\sum_{j\leq N}\lambda_j(A) \geq \frac{2\pi N^2}{|\omega|} + \frac{B^2}{2\pi}|\omega|m(1-m).$$

Recall that $m := \left\{\frac{2\pi N}{B|\omega|}\right\}$. Subtracting $N\Lambda$ from its both sides, we get

$$\sum_{j=1}^{N}(\Lambda - \lambda_j(A)) \leq f(N),$$

where $f : \mathbb{R}_+ \to \mathbb{R}$,

$$f(z) := z\Lambda - \frac{2\pi z^2}{|\omega|} - \frac{B^2|\omega|}{2\pi}\left\{\frac{2\pi z}{B|\omega|}\right\}\left(1 - \left\{\frac{2\pi z}{B|\omega|}\right\}\right).$$

We are going to investigate the function $f(z)$ on the intervals

$$\frac{B|\omega|k}{2\pi} \leq z < \frac{B|\omega|(k+1)}{2\pi}, \quad k = 0, 1, 2, \ldots$$

It is easy to check that $f'(z) = \Lambda - \frac{4\pi}{|\omega|} z - B + 2B \left\{ \frac{2\pi z}{B|\omega|} \right\}$, thus the extremum of f is achieved at the point z_0 satisfying

$$\Lambda - B - \frac{4\pi}{|\omega|} z_0 + 2B \left\{ \frac{2\pi z_0}{B|\omega|} \right\} = 0.$$

Denoting $x_0 := \frac{2\pi z_0}{B|\omega|}$, the condition reads

$$\Lambda - 2Bx_0 - B + 2B\{x_0\} = 0$$

giving

$$x_0 = \frac{\Lambda - B + 2B\{x_0\}}{2B}.$$

It yields the value of function f at z_0

$$f(z_0) = \frac{|\omega|(\Lambda^2 - B^2)}{8\pi}.$$

Furthermore, the values of f at the endpoints $\frac{Bk|\omega|}{2\pi}$, $k \in \mathbb{N}$ equal

$$f\left(\frac{Bk|\omega|}{2\pi}\right) = \frac{Bk|\omega|}{2\pi}(\Lambda - kB) \leq$$

$$\leq \frac{(\Lambda^2 - B^2)|\omega|}{8\pi} + \frac{(\Lambda - B)B|\omega|}{4\pi}\left\{\frac{\Lambda + B}{2B}\right\}.$$

Recall, that in the extremum point z_0 we have

$$f(z_0) = \frac{|\omega|(\Lambda^2 - B^2)}{8\pi}.$$

Hence

$$f(z) \leq \frac{(\Lambda^2 - B^2)|\omega|}{8\pi} + \frac{(\Lambda - B)B|\omega|}{4\pi}\left\{\frac{\Lambda + B}{2B}\right\}, \ \forall z \geq 0.$$

Furthermore, the values of f at the endpoints $\frac{Bk|\omega|}{2\pi}$, $k \in \mathbb{N}$ equal

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Thus, we arrive at

$$\sum_{j=1}^{N}(\Lambda-\lambda_j(A)) \leq f(N) \leq \frac{(\Lambda^2-B^2)|\omega|}{8\pi}+\frac{(\Lambda-B)B|\omega|}{4\pi}\left\{\frac{\Lambda+B}{2B}\right\}$$

and Theorem 2 is proved.