DYNAMICAL SYSTEMS THEORY OF PUBLIC OPINION

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Talk presented at Hradec Kralove Conference “Chaos and what it can reveal”, May 9th, 2017

PETR SEBA IN KOCHI, JAPAN

OR THE STORY OF TEA

- Green

- Black

MODELING DEMOCRATIC DEBATE

- How collective decisions are achieved?

- Majority principle ubiquitous from bee to human

  with some twist

- Interpret democracy as

  assertive minorities in search of majority support

  and try to build mathematical model

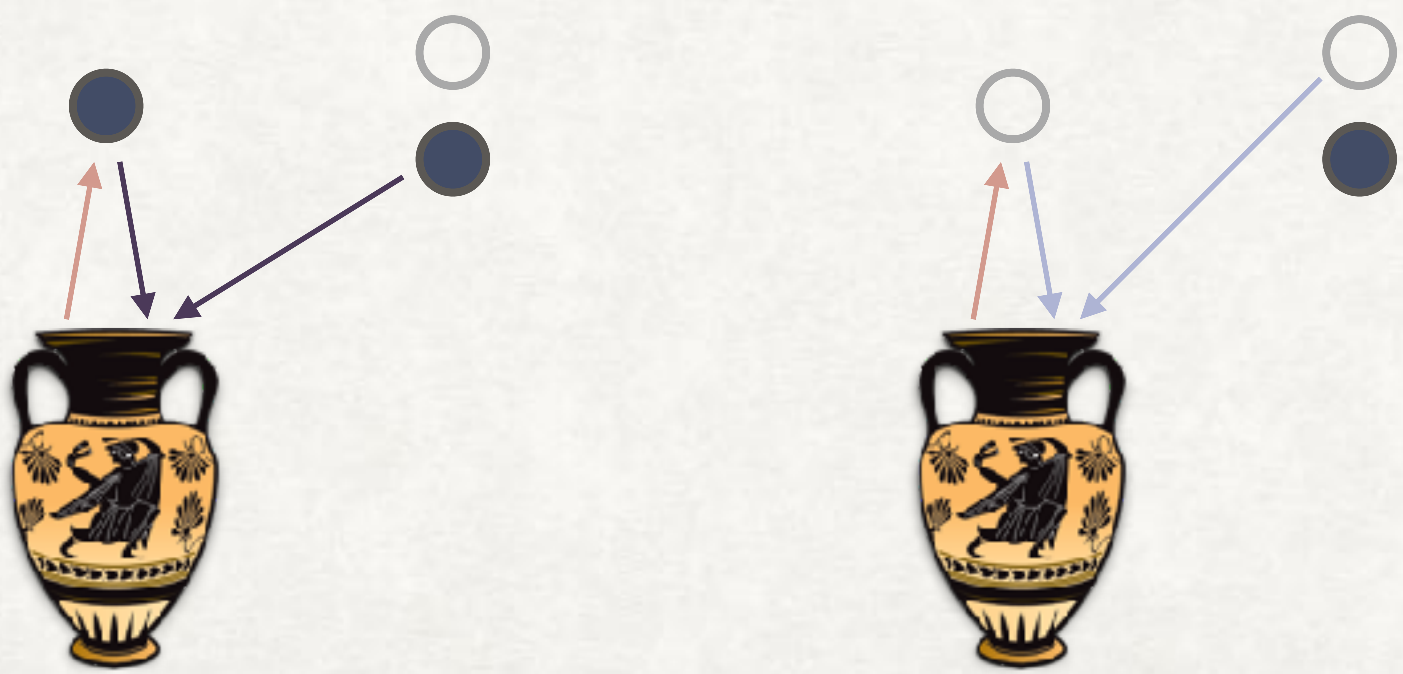
- Dynamical systems theory of political cycle obtained

POLYA URN

WHICH ONE?

POLYA URN

- t balls in two colors (m black/ t-m white) in an urn at time t
- At update t -> t+1, a ball randomly drawn,
  put back with <u>an additional ball with same color</u>
- What is the ratio of black balls $p_t = m/t$ at t ->infty ?



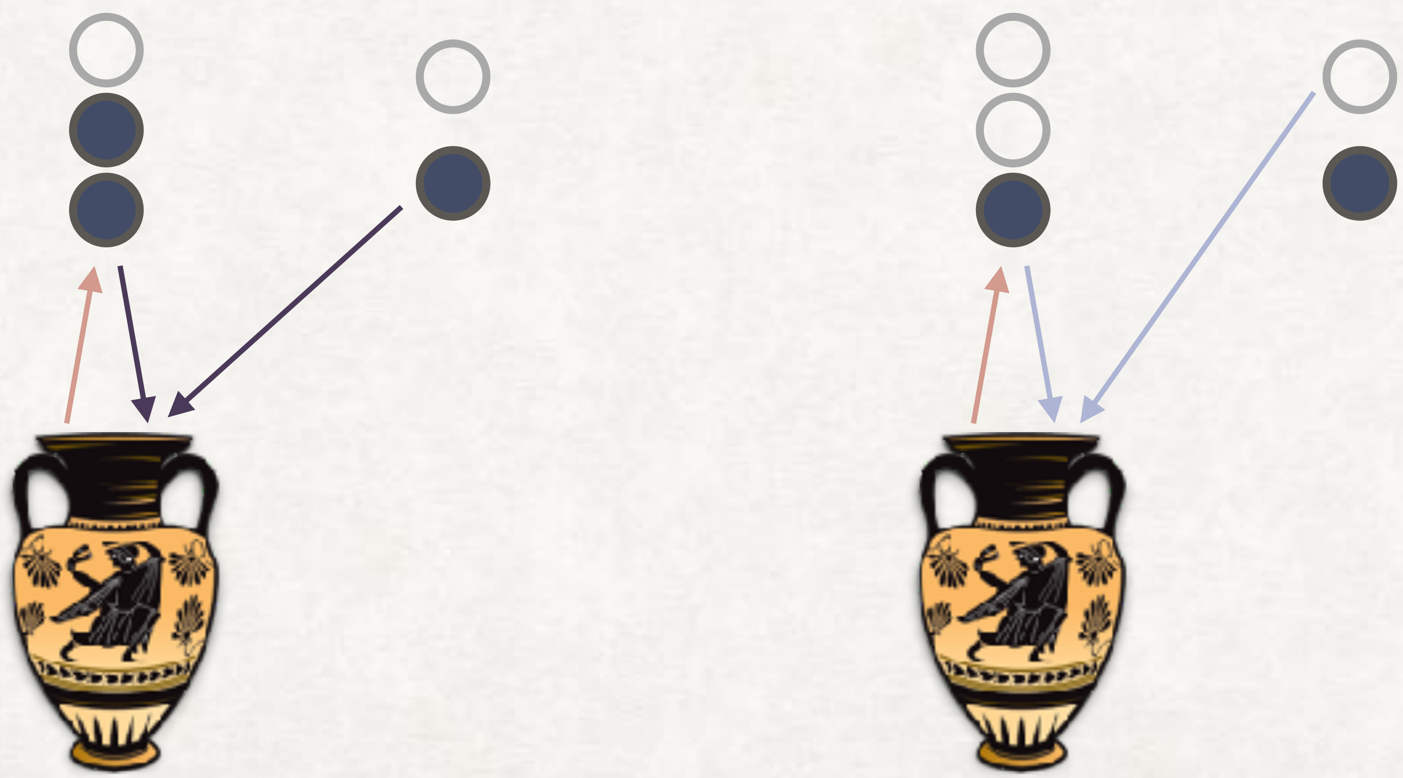
POLYA URN

EXTENDED

- At update $t \to t+1$, r ball randomly drawn,
  put back with an additional ball with majority color

- What is the ratio of black balls $p_t = m/t$ at $t \to \infty$?

$r=3$

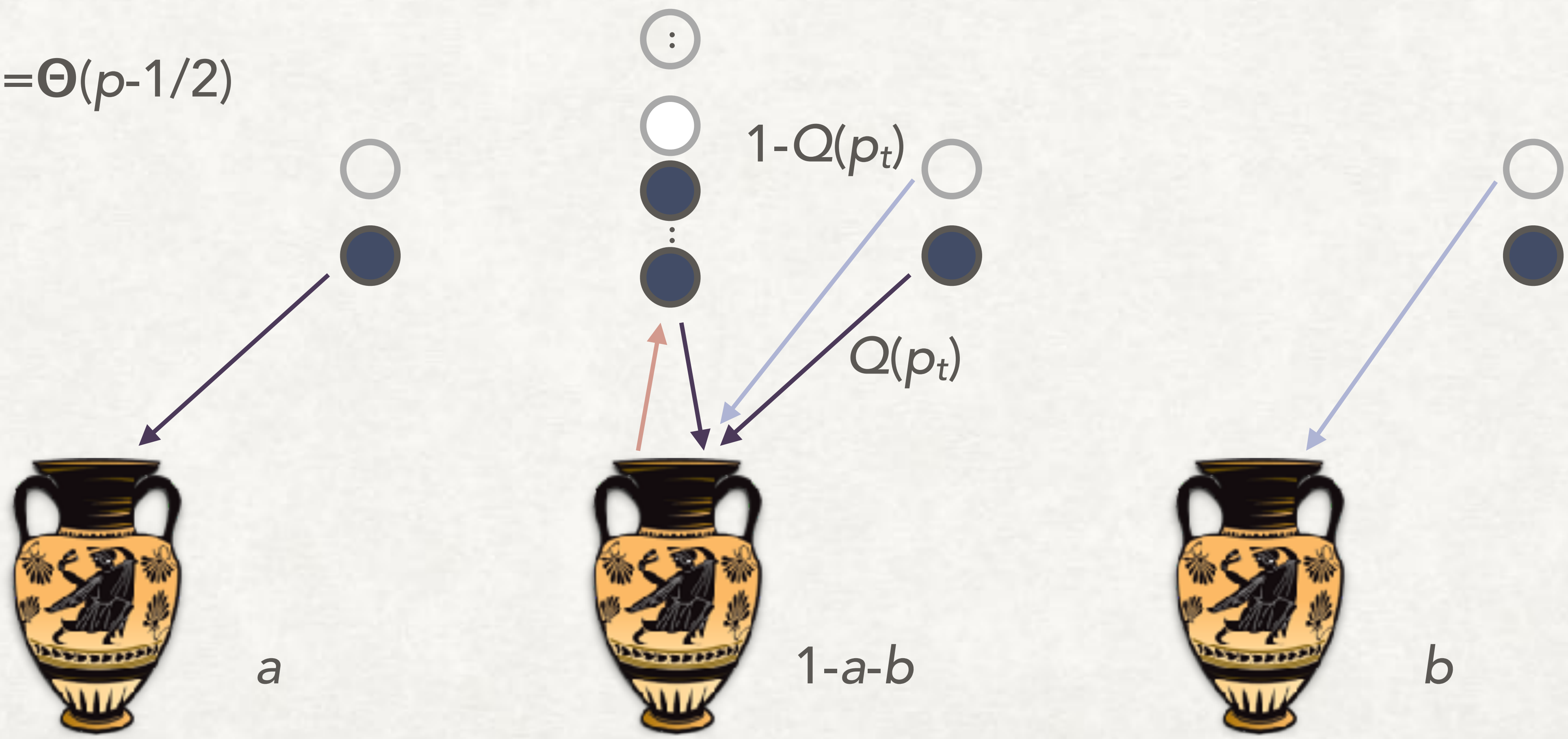


POLYA URN

HISAKADO-MORI MODEL

- At update t -> $t+1$
  — add a black / white ball with prob. a / b
  — with prob. $1\text{-}a\text{-}b$, count all t balls,
  add a black ball with prob. $Q(p_t)$, white with $1\text{-}Q(p_t)$

$Q(p)=\Theta(p\text{-}1/2)$

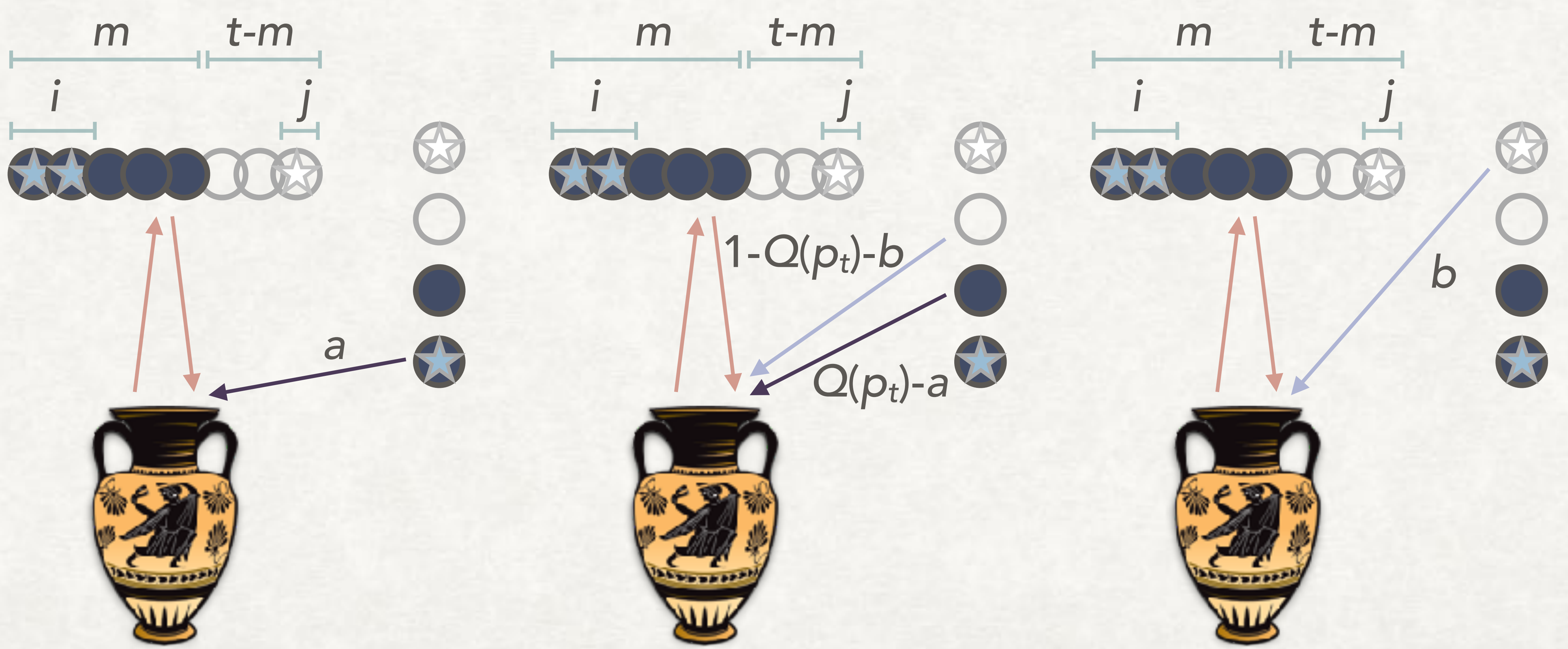


POLYA URN

HISAKADO-MORI EXPRESSED BY INFLEXIBLE HARD BALLS

- At update $t \to t+1$ $\qquad Q(p)=\Theta(p-1/2), \quad p_t=m/t$

— count all t balls, add a ball with <u>majority color</u>, except…
— if i hard-black balls found, <u>add a hard-black</u> with prob. $a = i/t$
— if j hard-white balls found, <u>add a hard-white</u> with prob. $b = j/t$



POLYA URN

MODELING THE DYNAMICS OF ASSERTIVE MINORITY

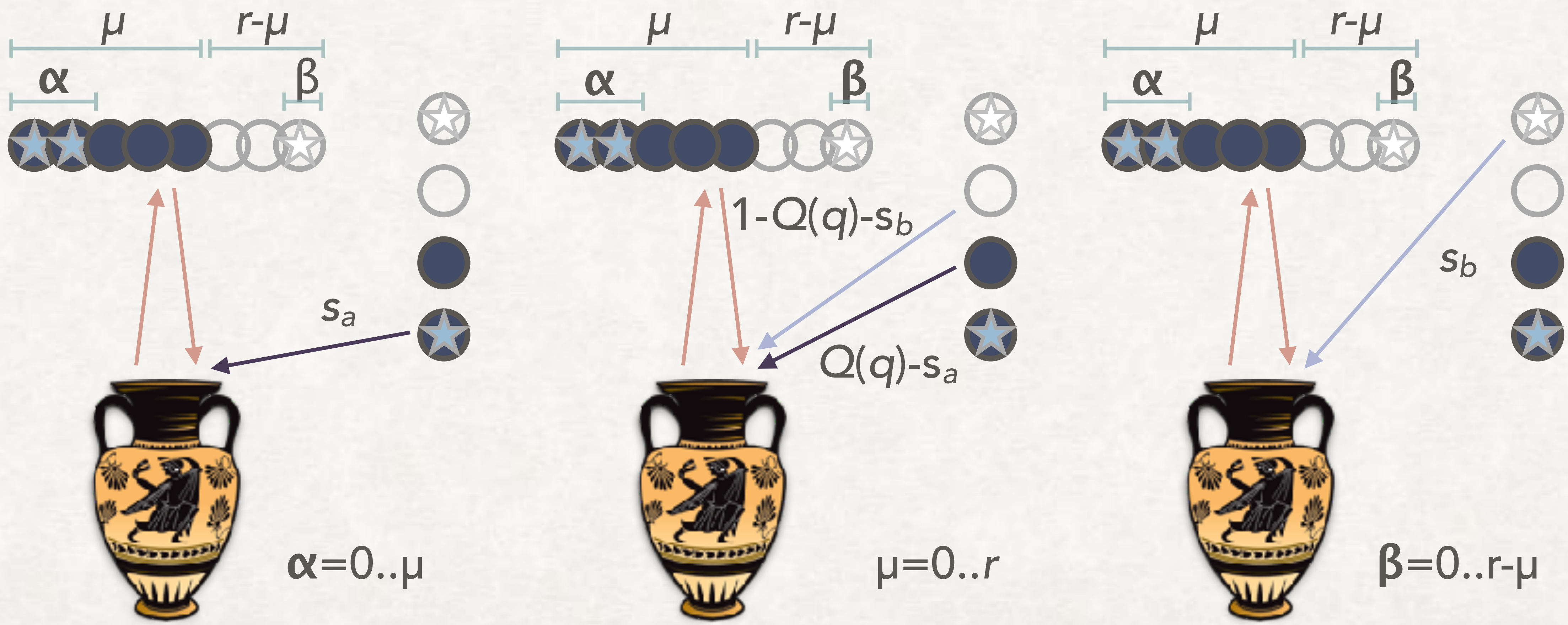
- At update t -> $t+1$ $Q(q)=\Theta(q-1/2)$, $q=\mu/r$

— sample r balls, add a ball with majority color, except…

— if α hard-black balls found, add a hard-black with $s_a=(1+f_\pm)\alpha/r$

— if β hard-white balls found, add a hard-white with $s_b=(1+g_\pm)\beta/r$

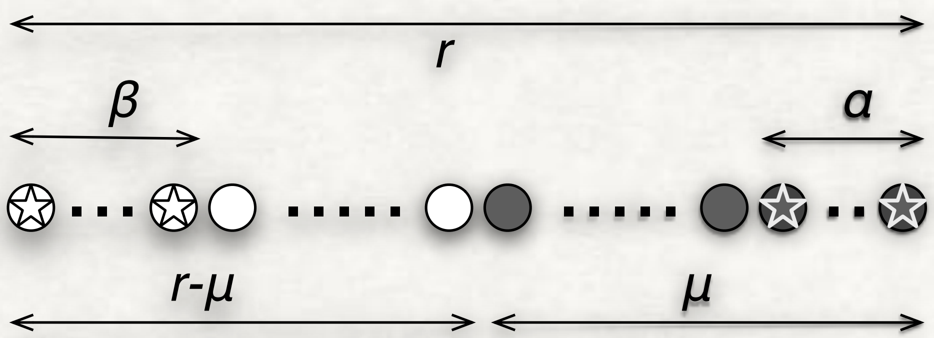
±: black/white majority

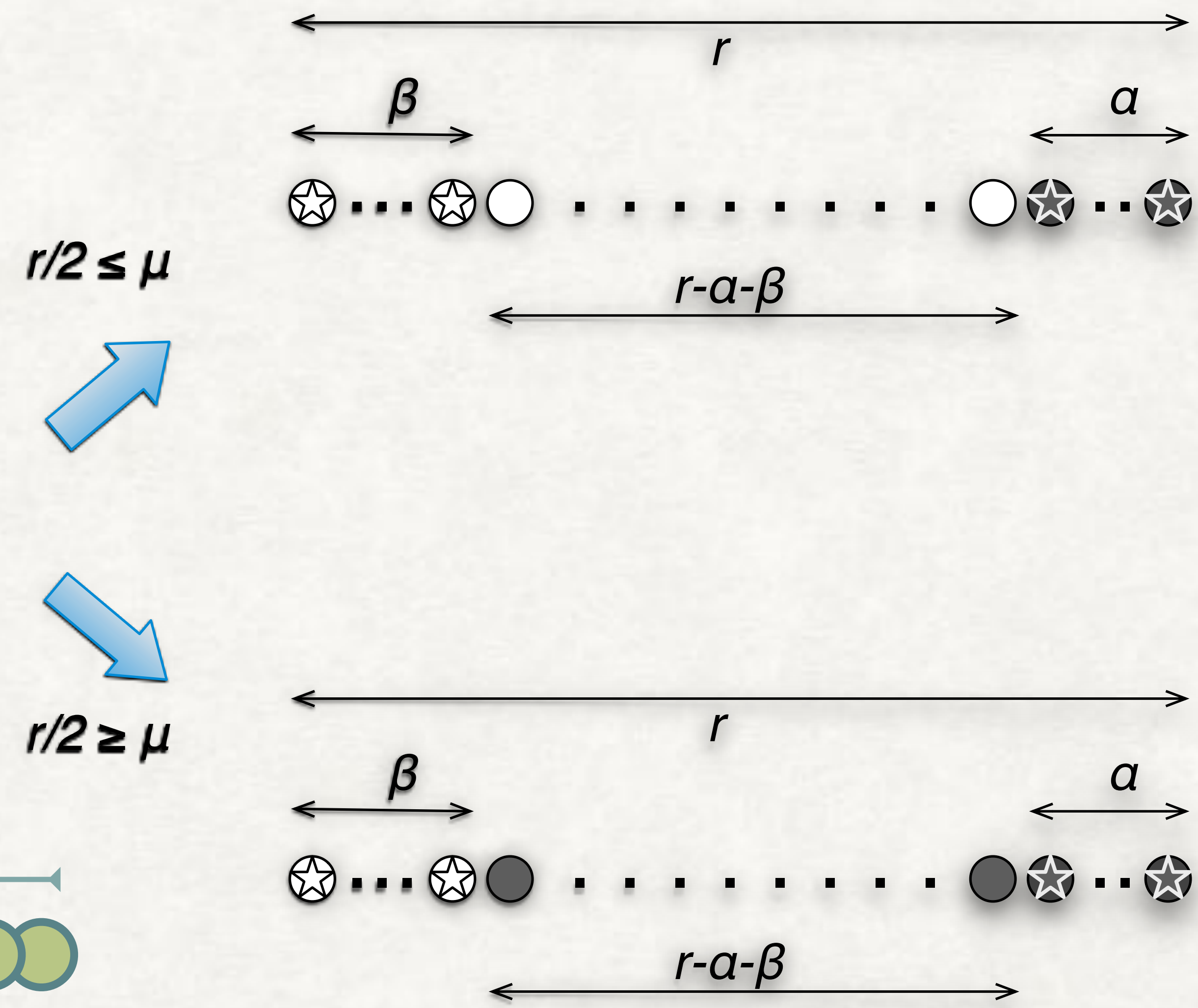


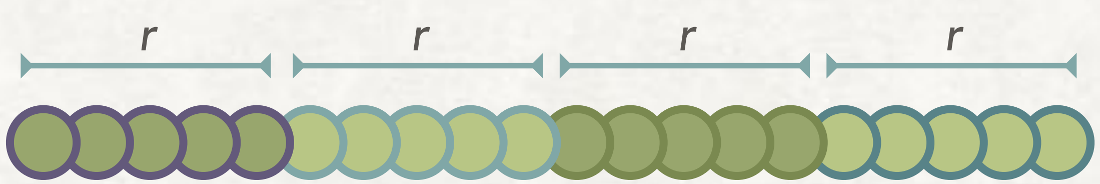
“DYNAMICAL” OPINION DYNAMICS

OUR MODEL AS EXTENDED GALAM MODEL

- Two-state agents evolving by *group-majority rule* (size *r*)
  with the presence of
  inflexible agents







EXTREMISTS AND MODERATES

EBB AND FLOW

- Committed few (extremists) drives political movement
- Extremists thrive in hostile environment
- Extremists normally lose their edge after success (moderates tend to suppress them in dominance)
- —> Increase/decrease rate of hard-black
  in friendly environ $(1+f_+) < 1$; in hostile environ $(1+f_-) > 1$
- —> Increase/decrease rate of hard-white
  in friendly environ $(1+g_-) < 1$; in hostile environ $(1+g_+) > 1$

“DYNAMICAL” OPINION DYNAMICS

- Evolution equation for majority and assertive minorities

$$p_{t+1} = P_+^{(r)}(p_t, a_t, b_t)$$
$$a_{t+1} = P_A^{(r)}(p_t, a_t, b_t)$$
$$b_{t+1} = P_B^{(r)}(p_t, a_t, b_t)$$

- Increase/decrease rate of hard-black
  in friendly environ $(1+f_+) < 1$; in hostile environ $(1+f_-) > 1$

- Increase/decrease rate of hard-white
  in friendly environ $(1+g_-) < 1$; in hostile environ $(1+g_+) > 1$

- Hard-black/hard-white appearance in all white/black env: f_A / g_B

GENERAL FORMULA FOR OPINION UPDATE

FOR ARBITRARY GROUP SIZE

(generalization of Cheon-Galam 2017)

$$P_{+}^{(r)} = \sum_{\mu=0}^{r} P_{+}^{(r,\mu)}, \quad P_{A}^{(r)} = \sum_{\mu=0}^{r} P_{A}^{(r,\mu)}, \quad P_{B}^{(r)} = \sum_{\mu=0}^{r} P_{B}^{(r,\mu)}.$$

- μ < r/2

$$P_{+}^{(r,\mu)}(p,a,b;f_{-}) = \binom{r}{\mu} p^{\mu-1}(1-p)^{r-\mu} \cdot \frac{\mu}{r} a(1+f_{-}),$$

$$P_{A}^{(r,\mu)}(p,a;f_{-}) = \binom{r}{\mu} p^{\mu-1}(1-p)^{r-\mu} \cdot \frac{\mu}{r} a(1+f_{-}),$$

$$P_{B}^{(r,\mu)}(p,b;g_{j}) = \binom{r}{\mu} p^{\mu}(1-p)^{r-\mu-1} \cdot \frac{r-\mu}{r} b(1+g_{-}),$$

- μ > r/2

$$P_{+}^{(r,\mu)}(p,a,b;g_{+}) = \binom{r}{\mu} p^{\mu}(1-p)^{r-\mu} - \binom{r}{\mu} p^{\mu}(1-p)^{r-\mu-1} \cdot \frac{r-\mu}{r} b(1+g_{+}),$$

$$P_{A}^{(r,\mu)}(p,a;f_{+}) = \binom{r}{\mu} p^{\mu-1}(1-p)^{r-\mu} \cdot \frac{\mu}{r} a(1+f_{+}),$$

$$P_{B}^{(r,\mu)}(p,b;g_{+}) = \binom{r}{\mu} p^{\mu}(1-p)^{r-\mu-1} \cdot \frac{r-\mu}{r} b(1+g_{+}).$$

"DYNAMICAL" OPINION DYNAMICS

- Evolution equation for majority and assertive minorities

r=3

$$P_+^{(3)}(p,a,b;f,g) = 3p^2 - 2p^3 + (1+f_-)(1-p)^2 a - (1+g_+)p^2 b + \frac{1}{3} f_A (1-p-b)^3 - \frac{1}{3} g_B (p-a)^3$$

$$P_A^{(3)}(p_t,a_t,b_t;f,g) = \left\{1 + f_+ + (f_- - f_+)(1-p)^2\right\} a + \frac{1}{3} f_A (1-p-b)^3$$

$$P_B^{(3)}(p_t,a_t,b_t;f,g) = \left\{1 + g_- + (g_+ - g_-)p^2\right\} b + \frac{1}{3} g_B (p-a)^3$$

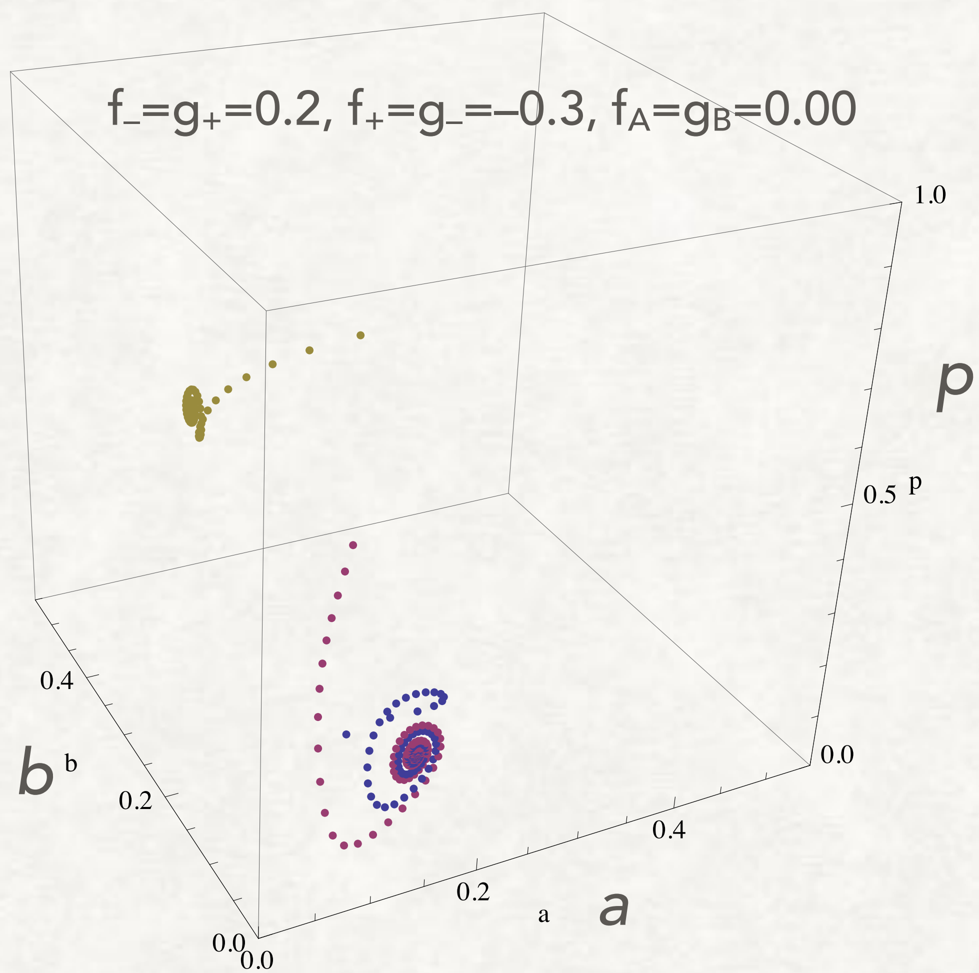
For full explicit expression for general r, see T.Cheon 2017 (in draft)

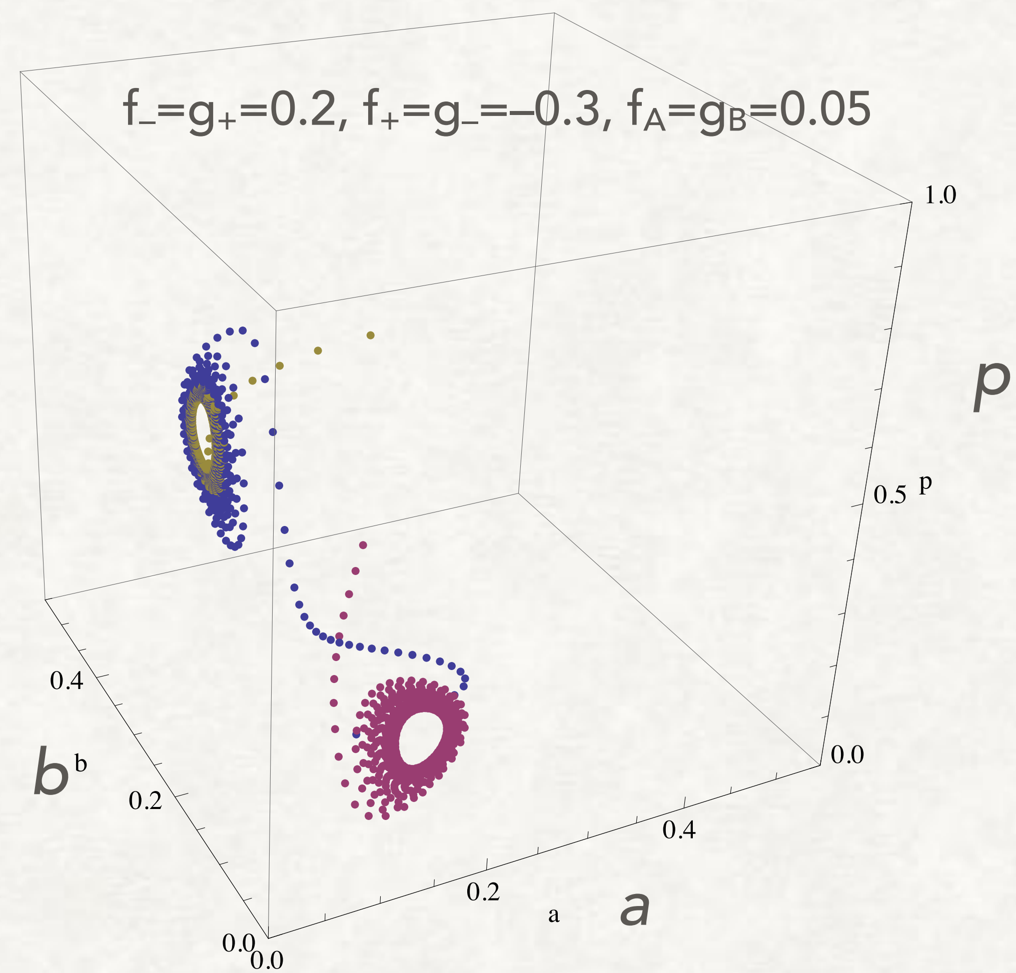
- Increase/decrease of hard-black
  in friendly environ $(1+f_+) < 1$; in hostile environ $(1+f_-) > 1$

- Increase/decrease of hard-white
  in friendly environ $(1+g_-) < 1$; in hostile environ $(1+g_+) > 1$

FIXED POINT AND LIMIT CYCLE

- numerics with r=3; Phase space trajectories



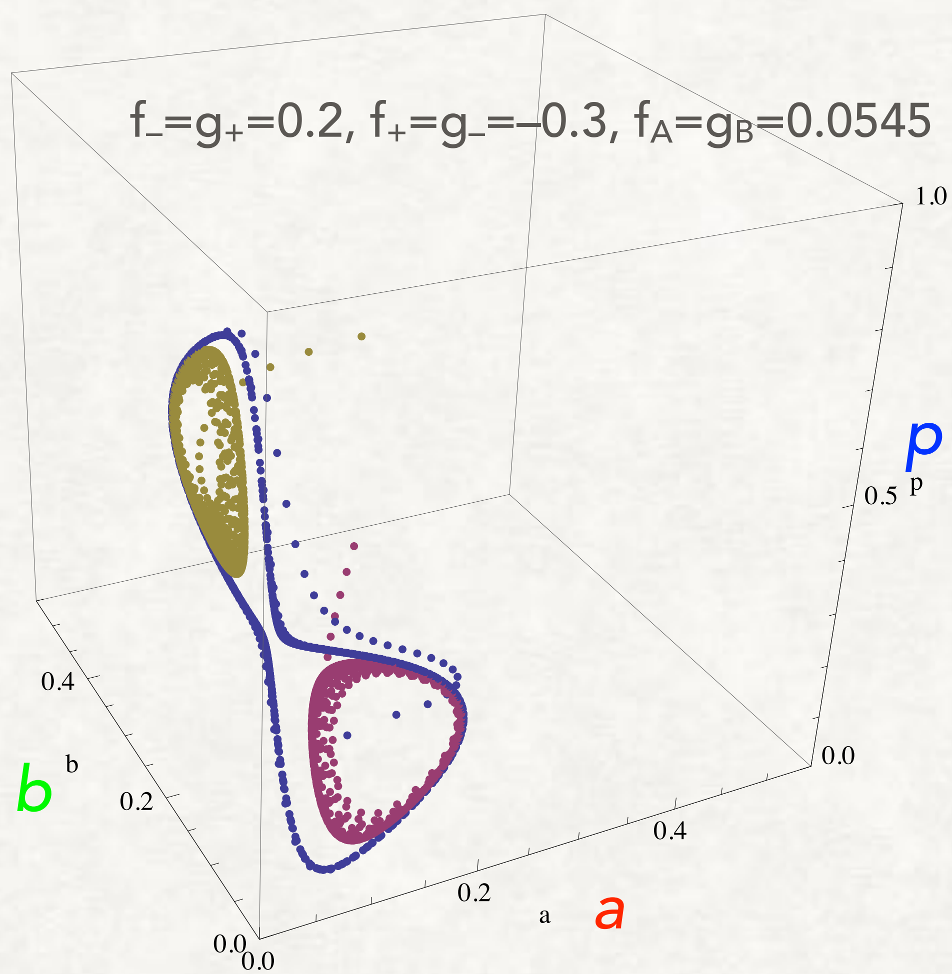


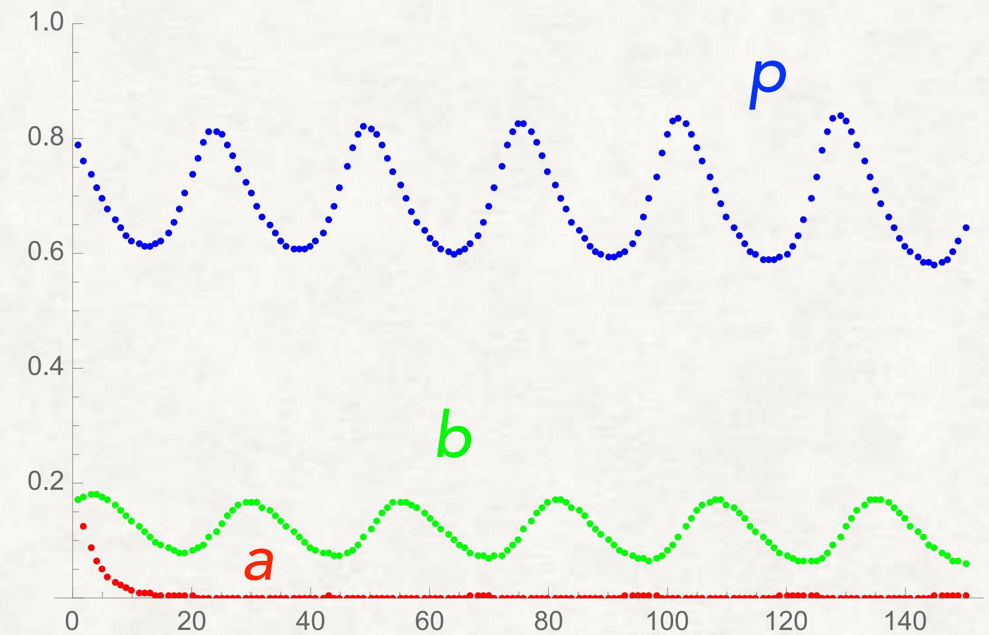
a^* (or b^*)~ $3-2\sqrt{2}$ 17% p^* (or $1-p^*$) ~ $(2-\sqrt{2})/2$ 29%

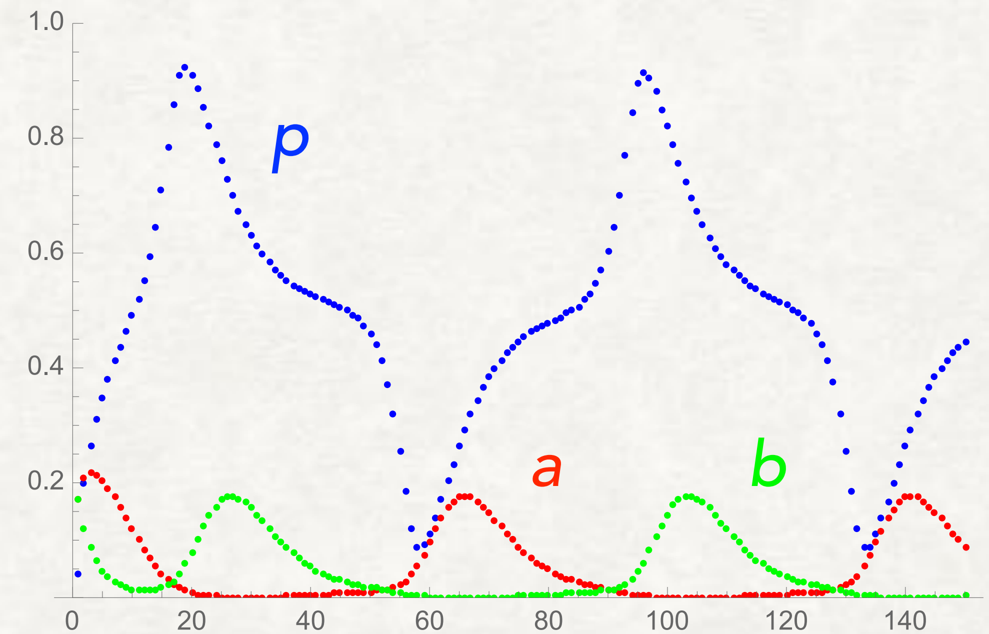
POLITICAL CYCLES

MAJORITY-ALTERNATING CYCLE

- numerics with r=3







a^* (or b^*)~ $3-2\sqrt{2}$ 17%

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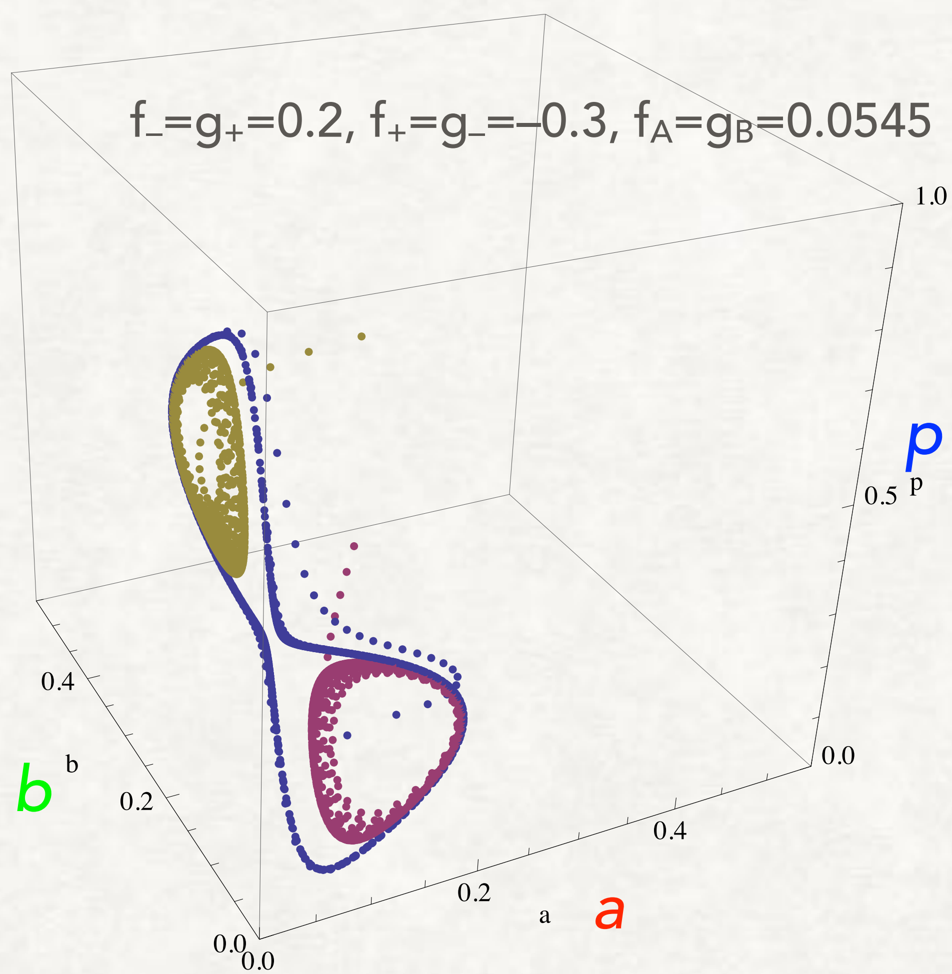
POLITICAL CYCLES

CYCLE WITHIN MINORITY

- numerics with r=3



a^* (or b^*)~ $3-2\sqrt{2}$ 17%

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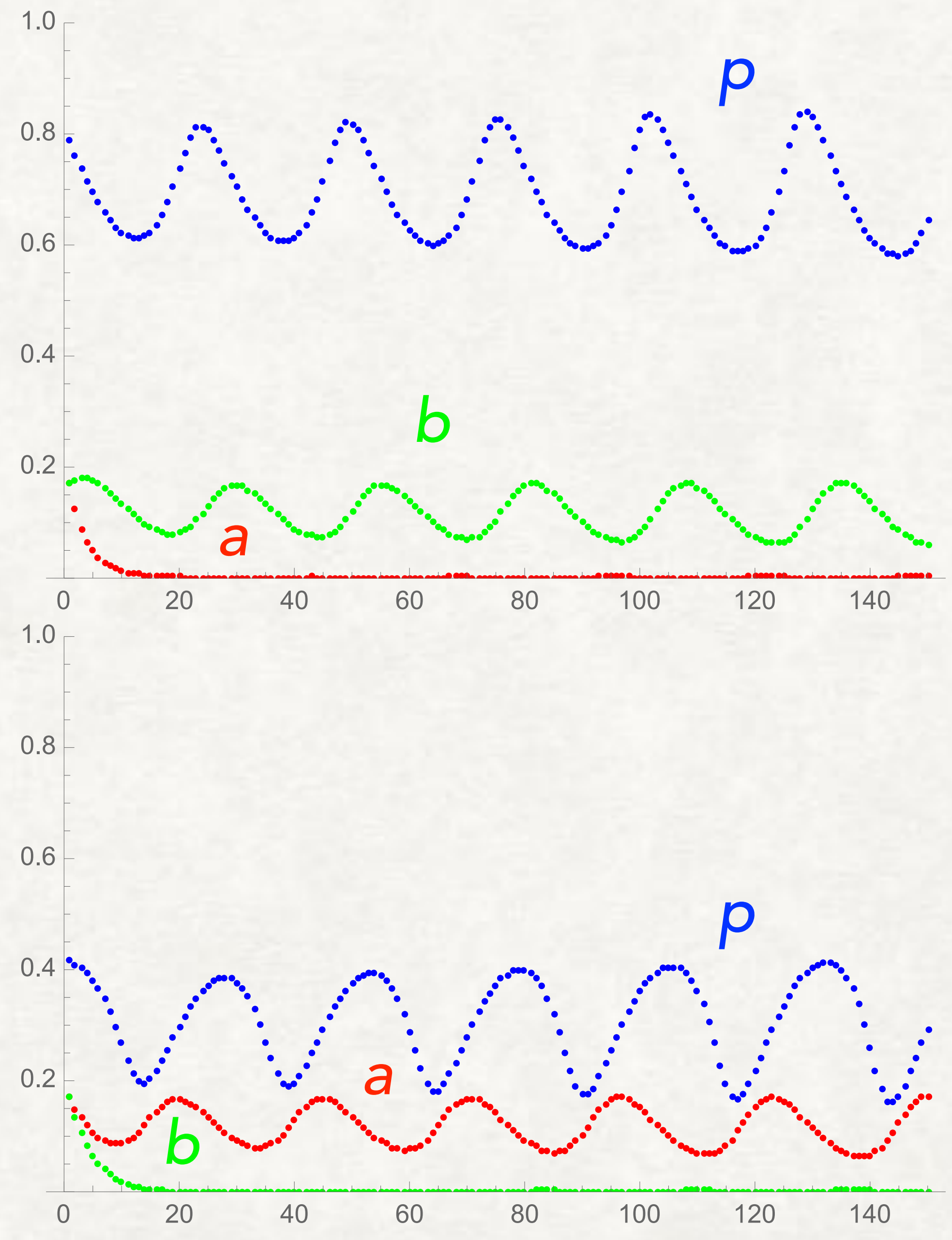


ON POLITICAL CYCLES

- Two fixed points with different majorities
  $\{p^*, a^*, b^*\} \sim \{0.3, 0.17, 0\}$
  $\{p^*, a^*, b^*\} \sim \{0.7, 0, 0.17\}$

30% minority with 17% extremists

What our model predicts

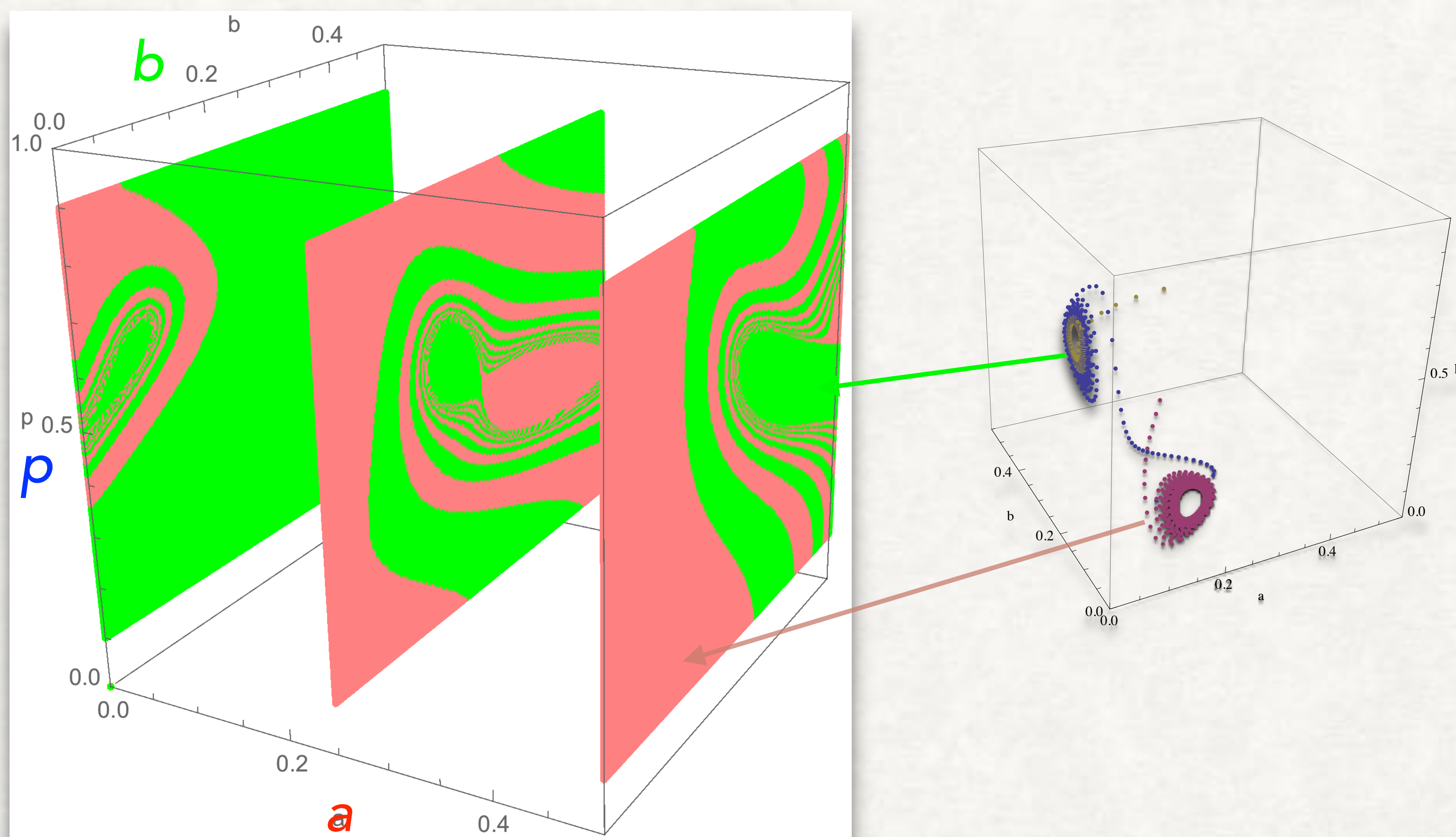
- Minority cycle with ebb and flow of extremists

- Majority changing cycle with double period — extremists driving the early phase of take-over, then disappears

COMPLEXITIES

BASINS OF ATTRACTION OF TWO MINORITY-CYCLES

- three sections of initial value space $\{p_0, a_0, b_0\}$ that results in black/ white minority cycles at "critical" parameters



$f_{-}=g_{+}=0.2$, $f_{+}=g_{-}=-0.46$, $f_A=g_B=0.0757$, $x=y=0.02$

SUMMARY

- Tea, green & black

- Dynamical systems theoretical model public opinion developed

- The theory unifies opinion dynamics models of Galam and Mori- Hisakado with analytical expressions in the form of Polya urn

- Existence of political cycles in the model discovered which may be capturing some aspect of real politics

  —> Toward the mathematical theory of politics!






Helen [redacted]
October 31,1936 - August 3, 2015


Dedicated to our great Tea Mater, Petr Seba, for the occasion of his 60th birthday.