



Some unusual spectra of periodic quantum graphs

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A talk at the conference **Chaos, and what it can reveal**

Hradec Králové, May 10, 2017

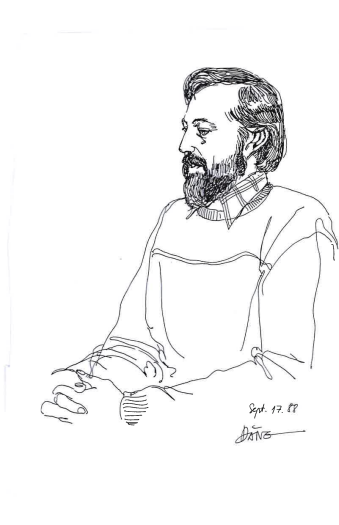
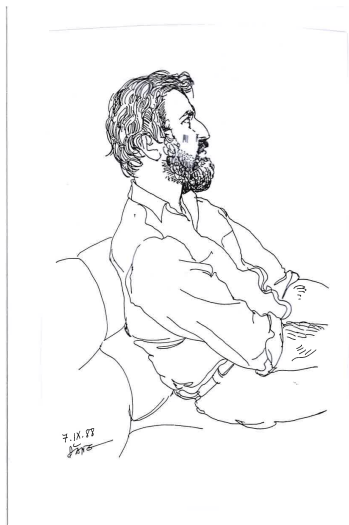
Do you recognize these two guys?



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This is to show that I know the man who brought us here for more than a half of his life – and little less of mine

What I am going to tell you



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Today I want to mention a few fresh results

Periodic quantum systems

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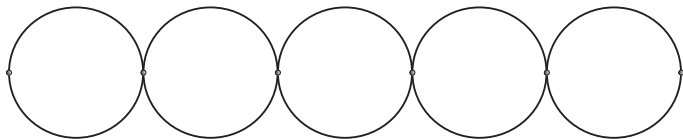
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Our starting observation is that if the system is a quantum graph, *nothing of that needs to be true!*

First example: chain graphs



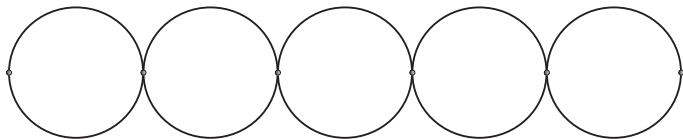
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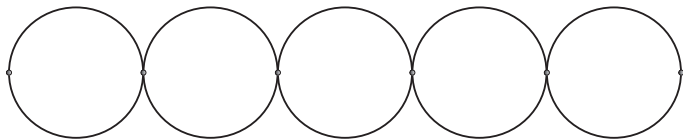


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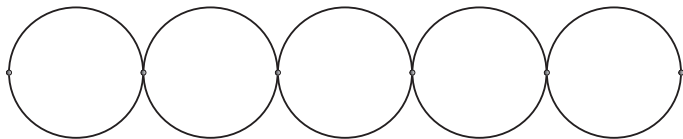
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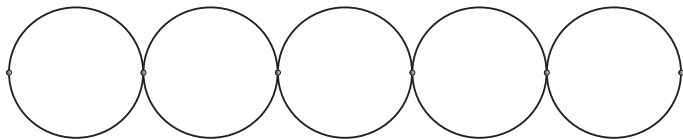
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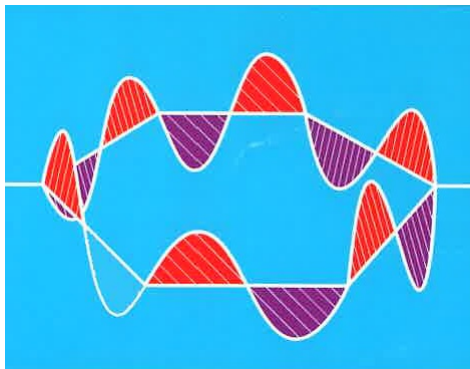


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Hence the spectrum is *not purely ac* and this trivial conclusion remains valid even if the chain loses its mirror symmetry but the 'upper' and 'lower' edge lengths are *rationally related*

Dirichlet eigenvalues are easy to understand



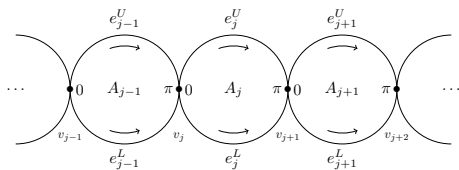
Courtesy: Peter Kuchment

It is also clear that quantum graphs can have compactly supported eigenfunctions

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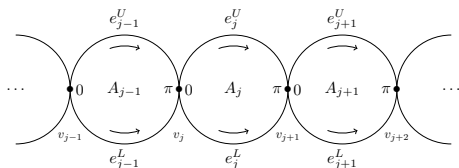
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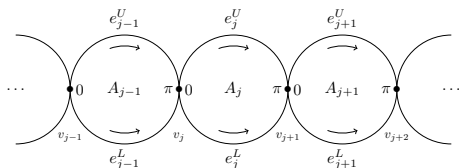


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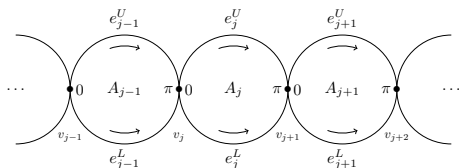
$$\psi_i(0) = \psi_j(0) =: \psi(0), \quad i, j \in \mathfrak{n}, \quad \sum_{i=1}^n \mathcal{D}\psi_i(0) = \alpha \psi(0),$$

where $\mathfrak{n} = \{1, 2, \dots, n\}$ is the index set numbering the edges – in our case $n = 4$ – and $\alpha \in \mathbb{R}$ is the coupling constant

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This is a particular case of the general conditions that make the operator self-adjoint [Kostykin-Schrader'03]

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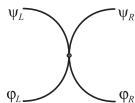
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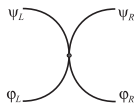
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- Without loss of generality we may suppose that the circumference of each ring is 2π , and as usual we employ units in which we have $\hbar = 2m = e = c = 1$, where e is electron charge (forget $\frac{e^2}{\hbar c} = \frac{1}{137}$)

Floquet-Bloch analysis of the fully periodic case



We write $\psi_L(x) = e^{-iAx}(C_L^+ e^{ikx} + C_L^- e^{-ikx})$ for $x \in [-\pi/2, 0]$ and energy $E := k^2 \neq 0$, and similarly for the other three components

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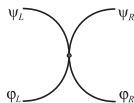
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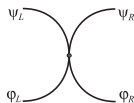
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with

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for any $k \in \mathbb{R} \cup i\mathbb{R} \setminus \{0\}$ and the discriminant equal to $D = 4(\xi(k)^2 - 1)$

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Apart from the cases $A - \frac{1}{2} \in \mathbb{Z}$ and $k \in \mathbb{N}$ we have $k^2 \in \sigma(-\Delta_\alpha)$ iff the condition $|\xi(k)| \leq 1$ is satisfied.

The fully periodic case, continued



Theorem (E-Manko'15)

Let $A \notin \mathbb{Z}$. If $A - \frac{1}{2} \in \mathbb{Z}$, then the spectrum of $-\Delta_\alpha$ consists of *two series of infinitely degenerate ev's* $\{k^2 \in \mathbb{R} : \xi(k) = 0\}$ and $\{k^2 \in \mathbb{R} : k \in \mathbb{N}\}$.

On the other hand, if $A - \frac{1}{2} \notin \mathbb{Z}$, the spectrum of $-\Delta_\alpha$ consists of *infinitely degenerate eigenvalues* k^2 with $k \in \mathbb{N}$, and *absolutely continuous spectral bands*. Each of these bands except the first one is contained in an interval $(n^2, (n+1)^2)$ with $n \in \mathbb{N}$. The first band is included in $(0, 1)$ if $\alpha > 4(|\cos A\pi| - 1)/\pi$, or it is negative if $\alpha < -4(|\cos A\pi| + 1)/\pi$, otherwise it contains the point $k^2 = 0$.

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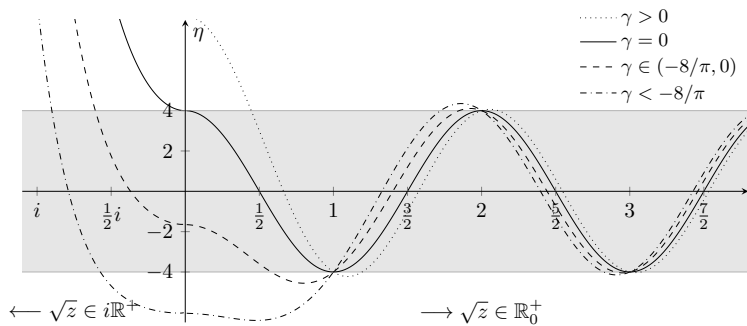
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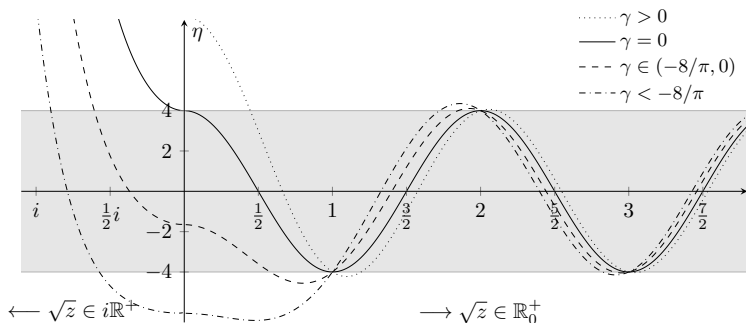
(b) In contrast to 'Dirichlet' eigenfunctions with one ring as an 'elementary cell', the 'other' eigenvalues arising for $A - \frac{1}{2} \in \mathbb{Z}$ are supported by *two adjacent rings*

In picture: determining the spectral bands



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For $A - \frac{1}{2} \notin \mathbb{Z}$ the situation is similar, just the width of the band changes to $4 \cos A\pi$, on the other hand, for $A - \frac{1}{2} \in \mathbb{Z}$ it *shrinks to a line*

Local perturbations



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A useful tool to treat them is to rephrase the problem as a system of *difference equation*

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$$\psi_{j+1}(k) + \psi_{j-1}(k) = \xi_j(k)\psi_j(k), \quad k \in \mathfrak{K},$$

where $\psi_j(k) := \psi(j\pi, k)$ and $\xi(k)$ was introduced above, ξ_j corresponding the coupling α_j . The two equations are intimately related.

Theorem

Let $\alpha_j \in \mathbb{R}$, then any solution $\begin{pmatrix} \psi(\cdot, k) \\ \varphi(\cdot, k) \end{pmatrix}$ with $k^2 \in \mathbb{R}$ and $k \in \mathfrak{K}$ satisfies the difference equation, and conversely, the latter defines via

$$\begin{pmatrix} \psi(x, k) \\ \varphi(x, k) \end{pmatrix} = e^{\mp iA(x-j\pi)} \left[\psi_j(k) \cos k(x - j\pi) + (\psi_{j+1}(k)e^{\pm iA\pi} - \psi_j(k) \cos k\pi) \frac{\sin k(x - j\pi)}{\sin k\pi} \right], \quad x \in (j\pi, (j+1)\pi),$$

solutions to the former satisfying the δ -coupling conditions. In addition, the former belongs to $L^p(\Gamma)$ if and only if $\{\psi_j(k)\}_{j \in \mathbb{Z}} \in \ell^p(\mathbb{Z})$, the claim being true for both $p \in \{2, \infty\}$.

Local perturbation examples



Consider first non-magnetic perturbations. We skip the theory referring to [Duclos-E-Turek'08, E-Manko'15] and show just examples of the results

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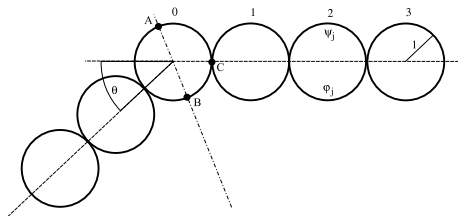
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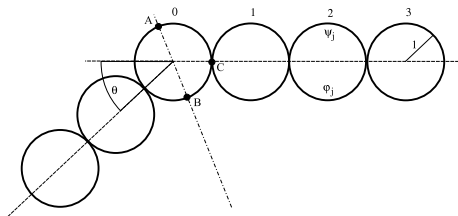
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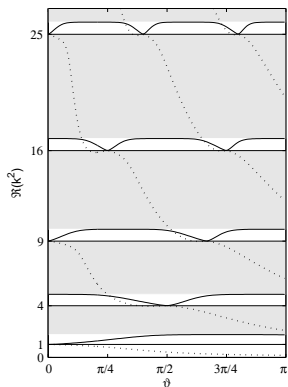
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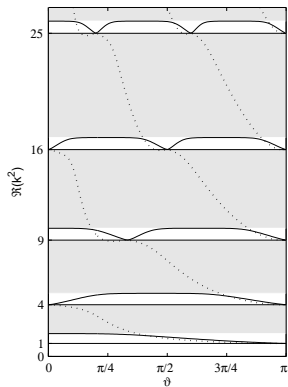
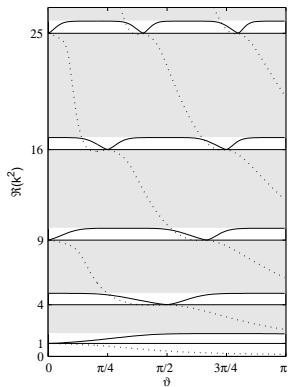


and ask how the spectrum depends on the angle ϑ . In this example we suppose that the magnetic field is absent

In picture: bent-chain spectrum for $\alpha = 3$

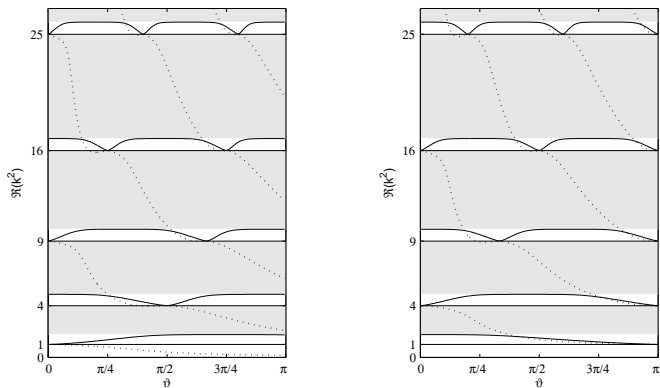


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for the even and odd part of the problem, respectively [Duclos-E-Turek'08]

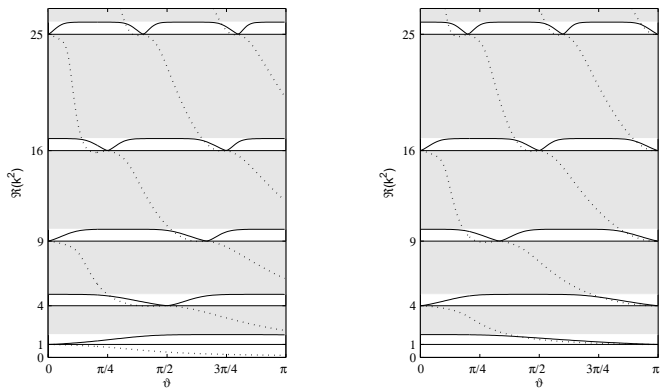
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Similar pictures we get for other values of α , the dotted lines in the figures mark (real values) of resonance positions

We see that the eigenvalues in gaps may be absent but only at rational values of ϑ and never simultaneously

Example: a single coupling constant changed

Let the couplings be $\{\dots, \alpha, \alpha + \gamma_1, \alpha, \dots\}$ and $A \notin \mathbb{Z}$, then we have



Proposition ([E-Manko'15])

Let $A \notin \mathbb{Z}$. The essential spectrum of $-\Delta_{\alpha+\gamma, A}$ coincides with that of $-\Delta_{\alpha}$. If $\gamma_1 < 0$ there is precisely one simple impurity state in every odd gap, on the other hand, for $\gamma_1 > 0$ there is precisely one simple impurity state in every even gap.

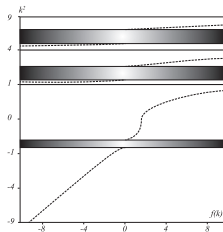
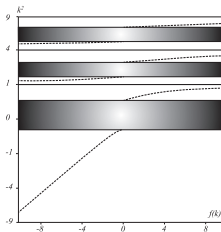
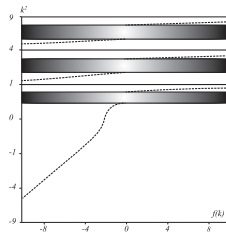
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The energy k^2 vs. $\gamma_1 = f(k)$ for $\cos A\pi = 0.6$ and the coupling strength
(i) $\alpha = 1$, (ii) $\alpha = -1$, (iii) $\alpha = -3$

More general duality

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What is important, the above *duality holds again*, with the difference relation being

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where $\psi_j(k) := \psi(x_j, k)$, and the reconstruction formula becomes

$$\begin{aligned} \begin{pmatrix} \psi(x, k) \\ \varphi(x, k) \end{pmatrix} &= e^{\mp i A_j (x - x_j)} \left[\psi_j(k) \cos k(x - x_j) \right. \\ & \left. + (\psi_{j+1}(k) e^{\pm i A_j \ell_j} - \psi_j(k) \cos k\ell_j) \frac{\sin k(x - x_j)}{\sin k\ell_j} \right], \quad x \in (x_j, x_{j+1}), \end{aligned}$$

Example again: a single flux altered



We suppose that the field is modified on a single ring, i.e.

$A = \{\dots, A, A_1, A \dots\}$, then we have a single simple eigenvalue in each gap provided [E-Manko'17]

$$\frac{|\cos A_1\pi|}{|\cos A\pi|} > 1,$$

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Can periodic graphs have “wilder” spectra?

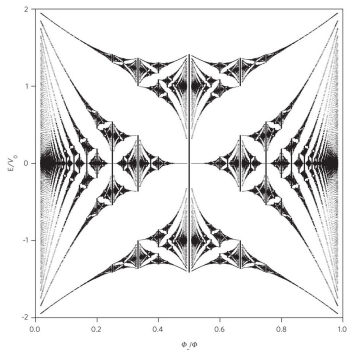
Let us first recall the picture everybody knows



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representing the spectrum of the difference operator associated with the *almost Mathieu equation*

$$u_{n+1} + u_{n-1} + 2\lambda \cos(2\pi(\omega + n\alpha))u_n = \epsilon u_n$$

for $\lambda = 1$, otherwise called *Harper equation*, as a function of α

Nice mathematics, but do such things exist?



Fractal nature of spectra for electron on a lattice in a homogeneous magnetic field was conjectured by [Azbel'64] but it caught the imagination only after Hofstadter made the structure visible

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Only recently an experimental realization of the original concept was achieved using a graphene lattice [Dean et al'13], [Ponomarenko'13]

Globally non-constant magnetic field



Our goal is now to investigate whether a similar effect can be seen in a 'one-dimensional' system. *The coupling constant will be in this part denoted γ !* To his aim we again employ duality

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We exclude the Dirichlet eigenvalues, $\sigma_D = \{k^2 : k \in \mathbb{N}\}$, and introduce

$$s(x; z) = \begin{cases} \frac{\sin(x\sqrt{z})}{\sqrt{z}} & \text{for } z \neq 0, \\ x & \text{for } z = 0, \end{cases} \quad \text{and} \quad c(x; z) = \cos(x\sqrt{z})$$

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Theorem (after Pankrashkin'13)

For any interval $J \subset \mathbb{R} \setminus \sigma_D$, the operator $(H_{\gamma, A})_J$ is unitarily equivalent to the pre-image $\eta^{(-1)}((L_A)_{\eta(J)})$, where L_A is the operator on $\ell^2(\mathbb{Z})$ acting as $(L_A q\varphi)_j = 2 \cos(A_j \pi) \varphi_{j+1} + 2 \cos(A_{j-1} \pi) \varphi_{j-1}$ and

$$\eta(z) := \gamma s(\pi; z) + 2c(\pi; z) + 2s'(\pi; z)$$

Corollary

The spectrum of $-\Delta_{\gamma,A}$ is bounded from below and can be decomposed into the discrete set $\sigma_D = \{n^2 \mid n \in \mathbb{N}\}$ of infinitely degenerate eigenvalues and the part σ_{L_A} determined by L_A , $\sigma(-\Delta_{\gamma,A}) = \sigma_p \cup \sigma_{L_A}$, where σ_{L_A} can be written as the union

$$\sigma_{L_A} = \bigcup_{n=0}^{\infty} \sigma_n$$

with $\sigma_n = \eta^{(-1)}(\sigma(L_A)) \cap I_n$ for $n \geq 0$, $I_n = \eta^{(-1)}([-4, 4]) \cap (n^2, (n+1)^2)$ for $n > 0$, and $I_0 = \eta^{(-1)}([-4, 4]) \cap (-\infty, 1)$.

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When $\gamma \neq 0$, the spectrum has always gaps between the σ_n 's. For $\gamma > 0$, the spectrum is positive. For $\gamma < -8\pi$, the spectrum has a negative part and does not contain zero. Finally, $0 \in \sigma(-\Delta_{\gamma,A})$ holds if and only if $\gamma\pi + 4 \in \sigma(L_A)$.

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Pay attention: In general, the σ_n 's may *very different* from absolutely continuous spectral bands!

A linear field growth



Suppose now that $A_j = \alpha j + \theta$ holds for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. We denote the corresponding operator L_A by $L_{\alpha, \theta}$, i.e.

$$(L_{\alpha, \theta} \varphi)_j = 2 \cos(\pi(\alpha j + \theta)) \varphi_{j+1} + 2 \cos(\pi(\alpha j - \alpha + \theta)) \varphi_{j-1}$$

for all $j \in \mathbb{Z}$. The rational case, $\alpha = p/q$, is easily dealt with.

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Assume that $\alpha = p/q$, where p and q are relatively prime. Then

(a) If $\alpha j + \theta + \frac{1}{2} \notin \mathbb{Z}$ for all $j = 0, \dots, q-1$, then $L_{\alpha, \theta}$ has purely ac spectrum that consists of q closed intervals possibly touching at the endpoints. In particular, $\sigma(L_{\alpha, \theta}) = [-4|\cos(\pi\theta)|, 4|\cos(\pi\theta)|]$ holds if $q = 1$.

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(b) If $\alpha j + \theta + \frac{1}{2} \in \mathbb{Z}$ for some $j = 0, \dots, q-1$, then the spectrum of $L_{\alpha, \theta}$ is of pure point type consisting of q distinct eigenvalues of infinite degeneracy. In particular, $\sigma(L_{\alpha, \theta}) = \{0\}$ holds if $q = 1$.

An irrational slope



On the other hand, if $\alpha \notin \mathbb{Q}$ the spectrum of $L_{\alpha,\theta}$ is closely related to that of the almost Mathieu operator $H_{\alpha,\lambda,\theta}$ in the critical situation, $\lambda = 2$, acting as

$$(H_{\alpha,\theta,\lambda}\varphi)_j = \varphi_{j+1} + \varphi_{j-1} + \lambda \cos(2\pi\alpha j + \theta)\varphi_j$$

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Combining all these results we can describe the spectrum of our original operator in case the magnetic field varies linearly along the chain

The linear-field spectrum



Theorem (E-Vařata'17)

Let $A_j = \alpha j + \theta$ for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. Then for the spectrum $\sigma(-\Delta_{\gamma, A})$ the following holds:

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- (b) If $\alpha = p/q$ with p and q relatively prime, $\alpha j + \theta + \frac{1}{2} \notin \mathbb{Z}$ for all $j = 0, \dots, q-1$ and assumptions of (a) do not hold, then $-\Delta_{\gamma,A}$ has *infinitely degenerate ev's* at the points of $\{n^2 \mid n \in \mathbb{N}\}$ and *an ac part* of the spectrum in each interval $(-\infty, 1)$ and $(n^2, (n+1)^2)$, $n \in \mathbb{N}$ consisting of q closed intervals possibly touching at the endpoints.

The linear-field spectrum



Theorem (E-Vařata'17)

Let $A_j = \alpha j + \theta$ for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. Then for the spectrum $\sigma(-\Delta_{\gamma,A})$ the following holds:

- (a) If $\alpha, \theta \in \mathbb{Z}$ and $\gamma = 0$, then $\sigma(-\Delta_{\gamma,A}) = \sigma_{ac}(-\Delta_{\gamma,A}) \cup \sigma_{pp}(-\Delta_{\gamma,A})$ where $\sigma_{ac}(-\Delta_{\gamma,A}) = [0, \infty)$ and $\sigma_{pp}(-\Delta_{\gamma,A}) = \{n^2 \mid n \in \mathbb{N}\}$.
- (b) If $\alpha = p/q$ with p and q relatively prime, $\alpha j + \theta + \frac{1}{2} \notin \mathbb{Z}$ for all $j = 0, \dots, q-1$ and assumptions of (a) do not hold, then $-\Delta_{\gamma,A}$ has *infinitely degenerate ev's* at the points of $\{n^2 \mid n \in \mathbb{N}\}$ and *an ac part* of the spectrum in each interval $(-\infty, 1)$ and $(n^2, (n+1)^2)$, $n \in \mathbb{N}$ consisting of q closed intervals possibly touching at the endpoints.
- (c) If $\alpha = p/q$, where p and q are relatively prime, and $\alpha j + \theta + \frac{1}{2} \in \mathbb{Z}$ for some $j = 0, \dots, q-1$, then the spectrum $-\Delta_{\gamma,A}$ is of *pure point type* and such that in each interval $(-\infty, 1)$ and $(n^2, (n+1)^2)$, $n \in \mathbb{N}$ there are exactly q distinct eigenvalues and the remaining eigenvalues form the set $\{n^2 \mid n \in \mathbb{N}\}$. All the eigenvalues are infinitely degenerate.

The linear-field spectrum, continued



Theorem (E-Vařata'17, cont'd)

(d) If $\alpha \notin \mathbb{Q}$, then $\sigma(-\Delta_{\gamma, A})$ does not depend on θ and it is a disjoint union of the isolated-point family $\{n^2 \mid n \in \mathbb{N}\}$ and *Cantor sets*, one inside each interval $(-\infty, 1)$ and $(n^2, (n+1)^2)$, $n \in \mathbb{N}$. Moreover, the *overall Lebesgue measure* of $\sigma(-\Delta_{\gamma, A})$ is zero.

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Using a fresh result of [Last-Shamis'16] we can also show

Proposition

Let $A_j = \alpha j + \theta$ for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. There exist a dense G_δ set of the slopes α for which, and all θ , the Hausdorff dimension

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Remark: If you regard a linear field *unphysical*, you may either view it as an *idealization* or to replace it a *quasiperiodic function* with the *same slope* leading to *the same result*.

Changing topic: graphs with a few gaps only

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Berkolaiko and Kuchment say that the situation with graphs is similar, however, they add immediately that *this is not a strict law* and illustrate this claim on resonant gaps created by a graph 'decoration', see also [Schenker-Aizenman'00]

The question: is it a 'law' after all?

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The answer depends on the vertex coupling. Recall that the standard coupling conditions

$$(U - I)\Psi + i(U + I)\Psi' = 0,$$

where Ψ , Ψ' are vectors of values and derivatives at the vertex, U is an $n \times n$ unitary matrix, where n is the vertex degree, decomposes into Dirichlet, Neumann, and Robin parts corresponding to eigenspaces of U with eigenvalues -1 , 1 , and the rest, respectively; if the latter is absent we call such a coupling *scale-invariant*

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Theorem ([E-Turek'17])

An infinite periodic quantum graph does not belong to the Bethe–Sommerfeld class if the couplings at its vertices are scale-invariant.

Proof idea



The spectrum is determined by *secular equation* [B-K'13]: we define

$$F(k; \vec{\vartheta}) := \det \left(\mathbf{I} - e^{i(\mathbf{A} + k\mathbf{L})} \mathbf{S}(k) \right),$$

where the $2E \times 2E$ matrices \mathbf{A} , \mathbf{L} , and \mathbf{S} are as follows: the diagonal matrix \mathbf{L} is given by the lengths of the directed edges (bonds) of Γ ,

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We note that $F(k; \vec{\vartheta})$ depends on $\vec{\vartheta}$ and $(kl_0, kl_1, \dots, kl_d)$, where $\{\ell_0, \ell_1, \dots, \ell_d\}$, $d + 1 \leq E$ are the mutually different edge lengths of Γ . If the ℓ 's are rationally related, the function is *periodic* in k , hence if there is a gap, there are *infinitely many of them*

Proof idea, and an extension



If the lengths are *not* rationally related, their ratios can be *approximated* by *rationals* with an arbitrary precision.

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Recall next that the vertex conditions can be equivalently written as

$$\begin{pmatrix} I^{(r)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^* & I^{(n-r)} \end{pmatrix} \Psi$$

for certain r , S , and T , where $I^{(r)}$ is the identity matrix of order r ; the coupling is scale-invariant if and only if the square matrix $S = 0$

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We will consider two *associated* quantum graph Hamiltonians, H with the above vertex coupling, and H_0 where we replace S by zero

A result for this associated pair



Proposition ([E-Turek'17])

For the spectra $\sigma(H)$ and $\sigma(H_0)$ the following claims hold true:

- (i) If $\sigma(H_0)$ has an open gap, then $\sigma(H)$ has infinitely many gaps.*
- (ii) If the edge lengths are rationally dependent, then the gaps of $\sigma(H)$ asymptotically coincide with those of $\sigma(H_0)$.*

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$$S(k) = -I^{(n)} + 2 \begin{pmatrix} I^{(r)} \\ T^* \end{pmatrix} \left(I^{(r)} + TT^* - \frac{1}{ik} S \right)^{-1} \begin{pmatrix} I^{(r)} & T \end{pmatrix}$$

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The same is true for $\mathbf{S}(k)$, and as consequence, the spectrum at high energies is mostly determined by the scale-invariant part. \square

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Our next goal is to give an affirmative answer:

Theorem ([E-Turek'17])

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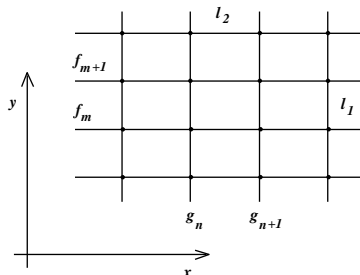


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As usual with existence claims, it is enough to demonstrate an example. With this aim we are going to revisit the model of a *rectangular lattice graph* introduced in [E'96, E-Gawlista'96]



Spectral condition



According to [E'96], a number $k^2 > 0$ belongs to a gap if and only if $k > 0$ satisfies the gap condition, which reads

$$\tan\left(\frac{ka}{2} - \frac{\pi}{2} \left\lfloor \frac{ka}{\pi} \right\rfloor\right) + \tan\left(\frac{kb}{2} - \frac{\pi}{2} \left\lfloor \frac{kb}{\pi} \right\rfloor\right) < \frac{\alpha}{2k} \quad \text{for } \alpha > 0$$

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where we denote the edge lengths ℓ_j , $j = 1, 2$, as a, b ; we neglect the Kirchhoff case, $\alpha = 0$, where $\sigma(H) = [0, \infty)$.

Note that for $\alpha < 0$ the spectrum extends to the negative part of the real axis and may have a gap there, which is not important here because there is not more than a single negative gap

What is known

The spectrum depends on the ratio $\theta = \frac{\ell_1}{\ell_2}$. If θ is rational, $\sigma(H)$ has infinitely many gaps unless $\alpha = 0$ in which case $\sigma(H) = [0, \infty)$



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we note that $\mu(\theta) = \mu(\theta^{-1})$

The golden mean situation

Let us start with the *golden mean*, $\phi = \frac{\sqrt{5}+1}{2} = [1; 1, 1, \dots]$, which can be regarded as the 'worst' irrational

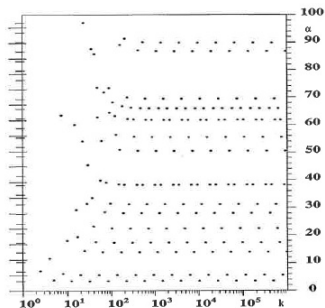


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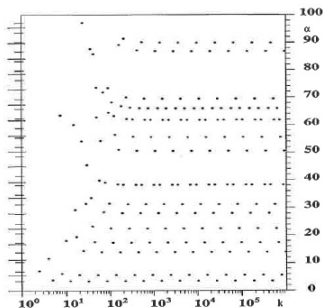


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Note that they approach the limit values *from above*, also that the series open at $\frac{\pi^2}{\sqrt{5ab}} \phi^{\pm 1/2} |n^2 - m^2 - nm|$, $n, m \in \mathbb{N}$ [E-Gawlista'96]

But a closer look shows a more complex picture



Theorem ([E-Turek'17])

Let $\frac{a}{b} = \phi = \frac{\sqrt{5}+1}{2}$, then the following claims are valid:

(i) If $\alpha > \frac{\pi^2}{\sqrt{5}a}$ or $\alpha \leq -\frac{\pi^2}{\sqrt{5}a}$, there are *infinitely many spectral gaps*.

(ii) If

$$-\frac{2\pi}{a} \tan\left(\frac{3-\sqrt{5}}{4}\pi\right) \leq \alpha \leq \frac{\pi^2}{\sqrt{5}a},$$

there are *no gaps* in the positive spectrum.

(iii) If

$$-\frac{\pi^2}{\sqrt{5}a} < \alpha < -\frac{2\pi}{a} \tan\left(\frac{3-\sqrt{5}}{4}\pi\right),$$

there is *a nonzero and finite number of gaps* in the positive spectrum.

But a closer look shows a more complex picture



Theorem ([E-Turek'17])

Let $\frac{a}{b} = \phi = \frac{\sqrt{5}+1}{2}$, then the following claims are valid:

(i) If $\alpha > \frac{\pi^2}{\sqrt{5}a}$ or $\alpha \leq -\frac{\pi^2}{\sqrt{5}a}$, there are *infinitely many spectral gaps*.

(ii) If

$$-\frac{2\pi}{a} \tan\left(\frac{3-\sqrt{5}}{4}\pi\right) \leq \alpha \leq \frac{\pi^2}{\sqrt{5}a},$$

there are *no gaps* in the positive spectrum.

(iii) If

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Corollary

The above theorem about the existence of BS graphs is valid.

More about this example



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For a given $N \in \mathbb{N}$, there are exactly N gaps in the positive spectrum if and only if α is chosen within the bounds

$$-\frac{2\pi(\phi^{2(N+1)} - \phi^{-2(N+1)})}{\sqrt{5}a} \tan\left(\frac{\pi}{2}\phi^{-2(N+1)}\right) \leq \alpha < -\frac{2\pi(\phi^{2N} - \phi^{-2N})}{\sqrt{5}a} \tan\left(\frac{\pi}{2}\phi^{-2N}\right).$$

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Note that the numbers $A_j := \frac{2\pi(\phi^{2j} - \phi^{-2j})}{\sqrt{5}} \tan\left(\frac{\pi}{2}\phi^{-2j}\right)$ form an increasing sequence the first element of which is $A_1 = 2\pi \tan\left(\frac{3-\sqrt{5}}{4}\pi\right)$ and

$$A_j < \frac{\pi^2}{\sqrt{5}} \quad \text{for all } j \in \mathbb{N}.$$

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Theorem ([E-Turek'17])

Let $\theta = \frac{a}{b}$ and define

$$\gamma_+ := \min \left\{ \inf_{m \in \mathbb{N}} \left\{ \frac{2m\pi}{a} \tan \left(\frac{\pi}{2} (m\theta^{-1} - \lfloor m\theta^{-1} \rfloor) \right) \right\}, \inf_{m \in \mathbb{N}} \left\{ \frac{2m\pi}{b} \tan \left(\frac{\pi}{2} (m\theta - \lfloor m\theta \rfloor) \right) \right\} \right\}$$

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and γ_- similarly with $\lfloor \cdot \rfloor$ replaced by $\lceil \cdot \rceil$. If the coupling constant α satisfies

$$\gamma_{\pm} < \pm\alpha < \frac{\pi^2}{\max\{a, b\}} \mu(\theta),$$

then there is *a nonzero and finite number of gaps* in the positive spectrum.

BS property does not need a definite sign of α



Proposition ([E-Turek'17])

Let the edge ratio be

$$\theta = \frac{2t^3 - 2t^2 - 1 + \sqrt{5}}{2(t^4 - t^3 + t^2 - t + 1)} \quad \text{for } t \in \mathbb{N}, t \geq 3;$$

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Note that the above number θ can be written as $\theta = \frac{t\phi+1}{(t^2+1)\phi+t}$ with $\phi = \frac{1+\sqrt{5}}{2}$, and moreover, the continued-fraction representation of θ is $[0; t, t, 1, 1, 1, 1, \dots]$. Furthermore, we have $\mu(\theta) = \mu(\phi) = \frac{1}{\sqrt{5}}$.

The talk was based on



[EM15] P.E., Stepan Manko: Spectra of magnetic chain graphs: coupling constant perturbations, *J. Phys. A: Math. Theor.* **48** (2015), 125302 (20pp)

[EM17] P.E., Stepan Manko: Spectral properties of magnetic chain graphs, *Ann. H. Poincaré* **18** (2017), 929–953.

[EV17] P.E., Daniel Vařata: Cantor spectra of magnetic chain graphs, *J. Phys. A: Math. Theor.* **50** (2017), 165201 (13pp)

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as well as the other papers mentioned in the course of the presentation.

It remains to say, albeit a bit belatedly



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Happy birthday, Petr!

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Happy birthday, Petr!
Biz hundred un tsvantsik!