The Brownian traveller on manifolds

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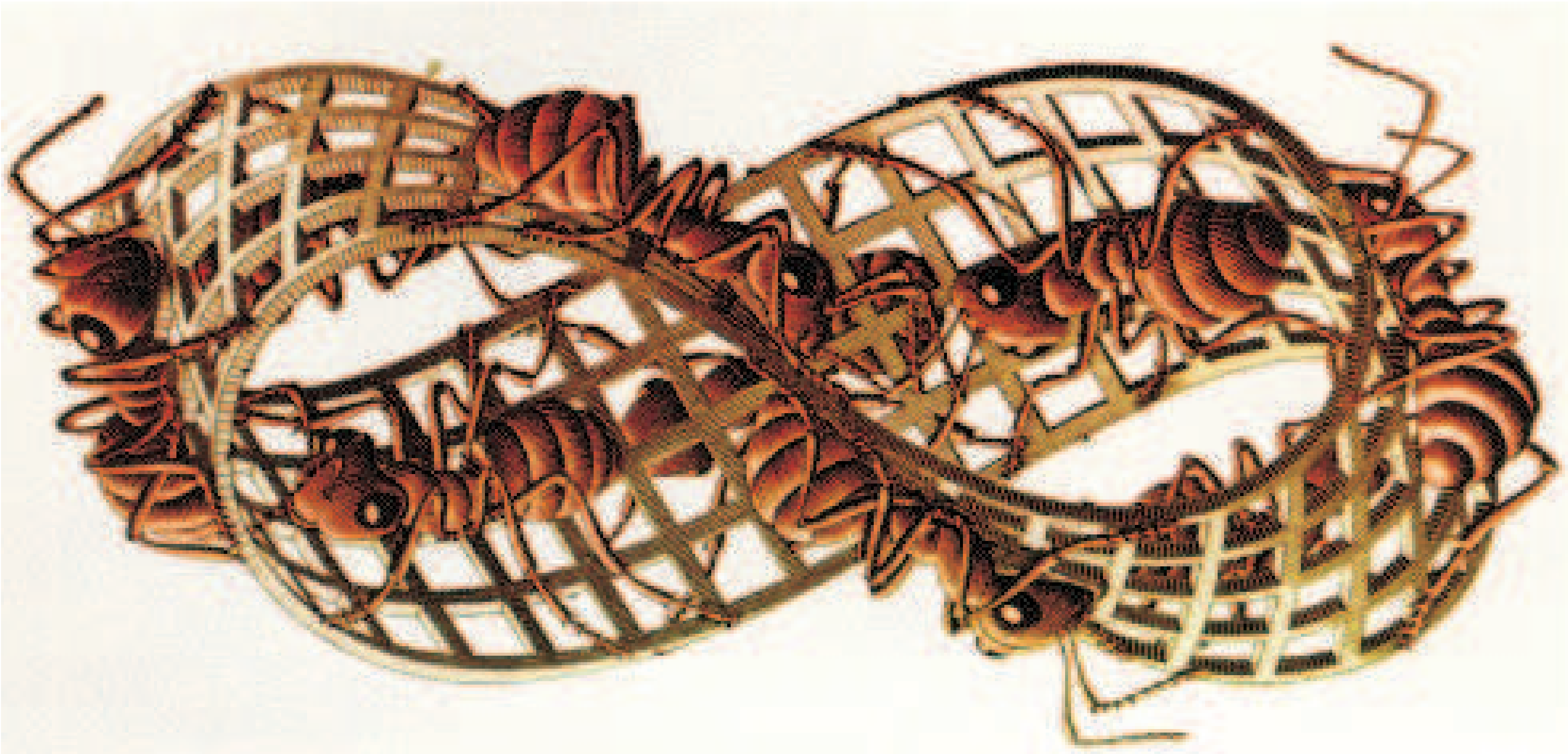
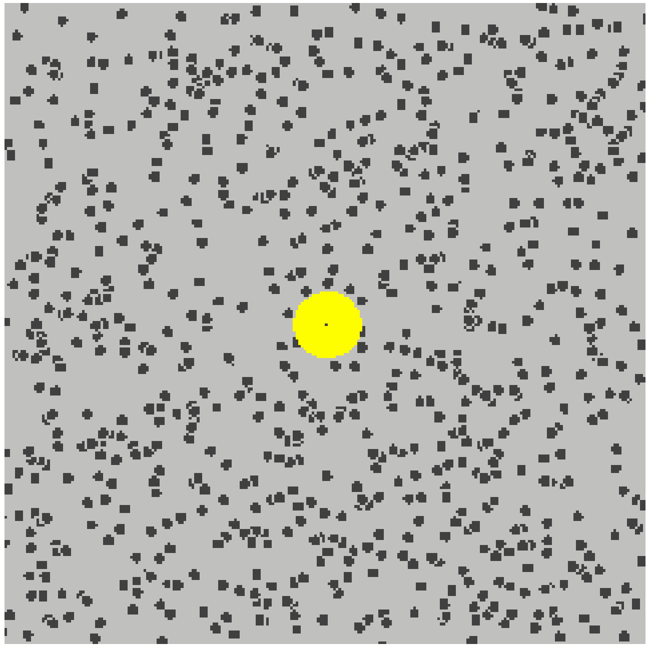


Dedicated to Petr Šeba on the occasion of his 60th birthday

Petr Šeba

Spectra, Algorithms and Data Analysis II, Hradec Králové, December 2006

The Brownian motion



Robert Brown

1773–1858

Albert Einstein

1879–1955

$$\frac{\partial p}{\partial \tau} - \Delta p = 0$$

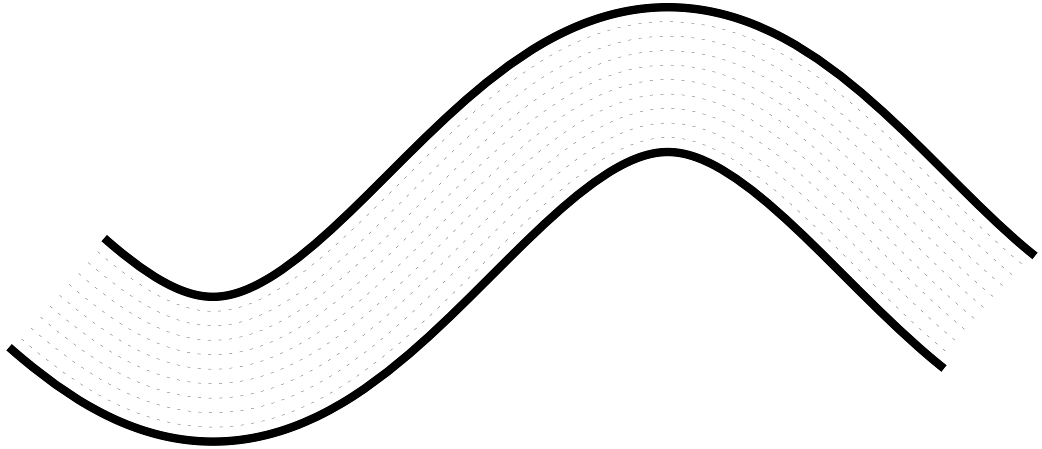
Jean Baptiste Perrin

1870–1942

Bound states in curved quantum waveguides

[Exner, Šeba 1989 (JMP)]

$\exists$ stationary solutions of $i\,\dfrac{\partial \Psi}{\partial \tau} = -\Delta \Psi$ in any locally *curved* Dirichlet strip $\Omega \subset \mathbb{R}^2$:



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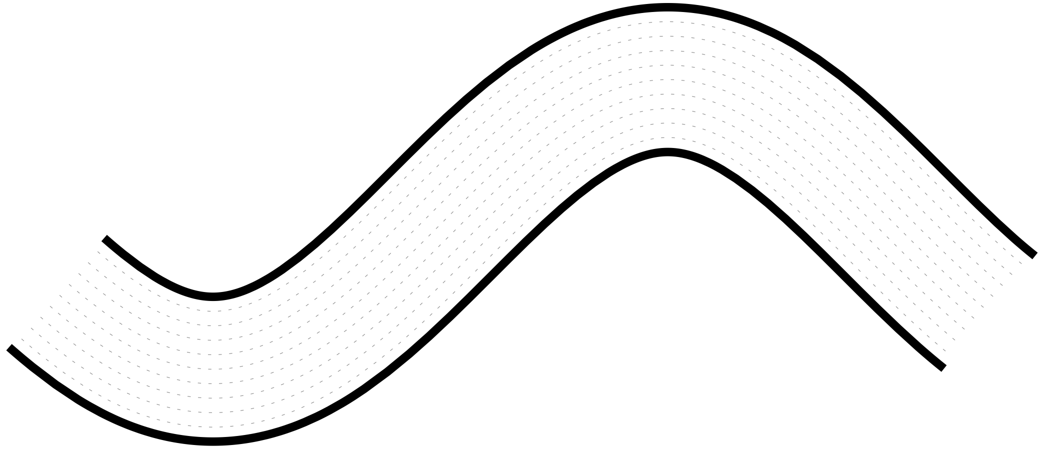
$\Psi(x,\tau) = e^{-i\lambda\tau}\,\psi(x)$

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in} \quad \Omega \\ \psi = 0 & \text{on} \quad \partial\Omega \end{cases}$$



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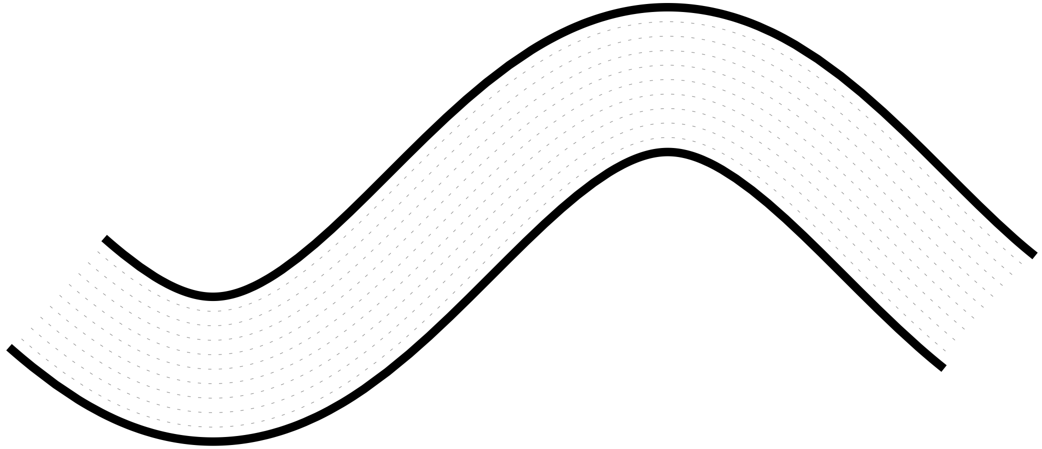

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$$\lambda_1 := \min \sigma(-\Delta) < \min \sigma_{\mathrm{ess}}(-\Delta) =: E_1$$

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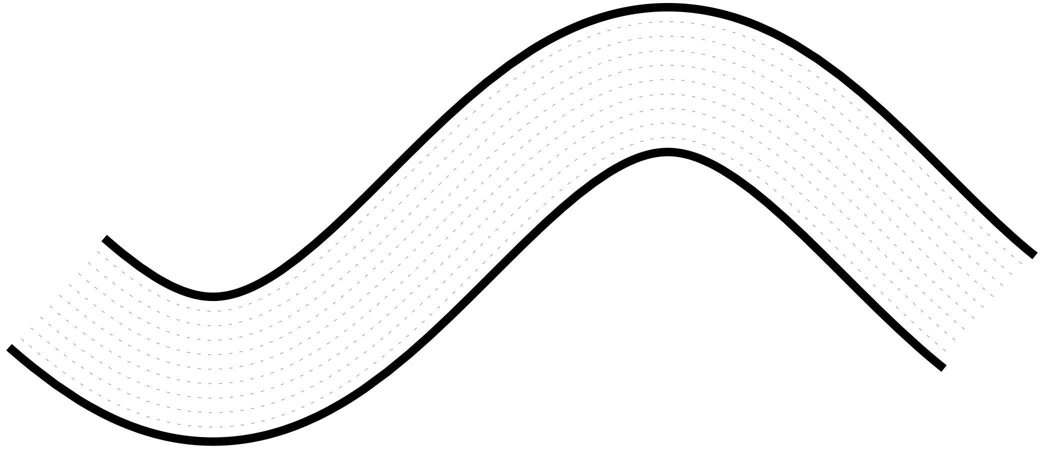

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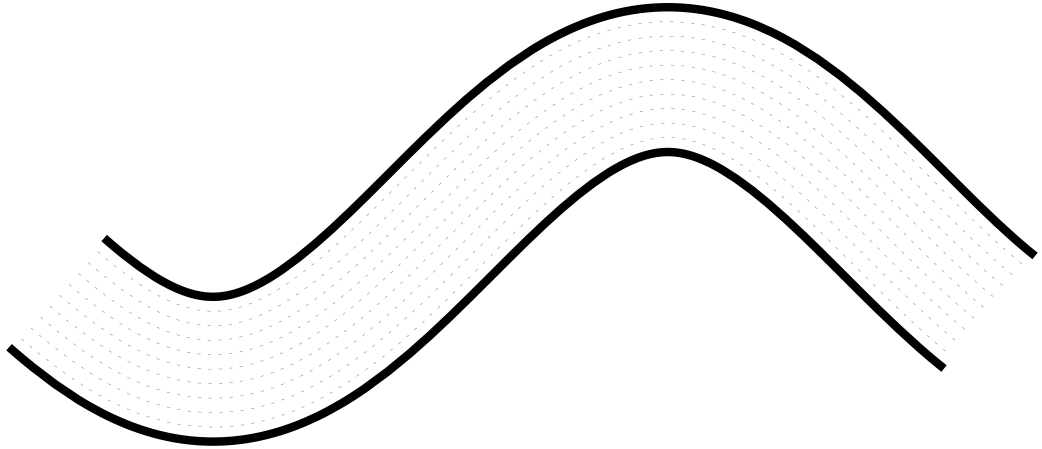

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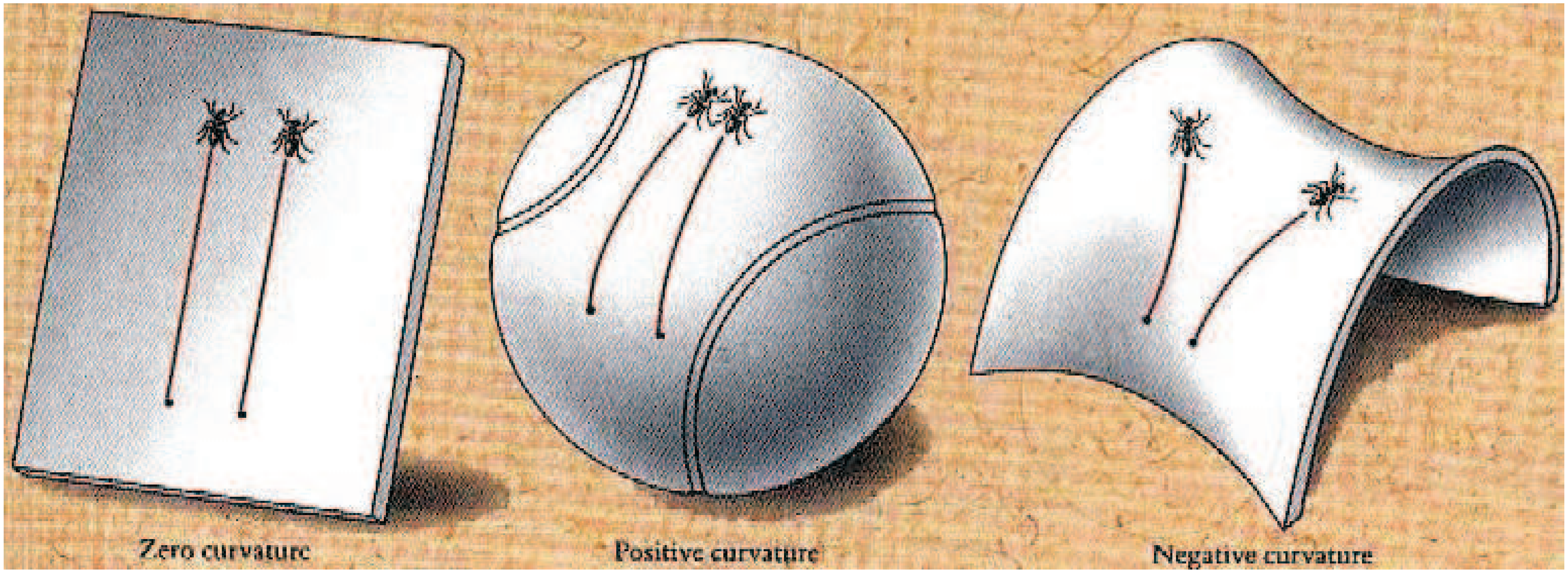
the Brownian particle lives longer in a curved strip

Which geometry is better to travel in ?



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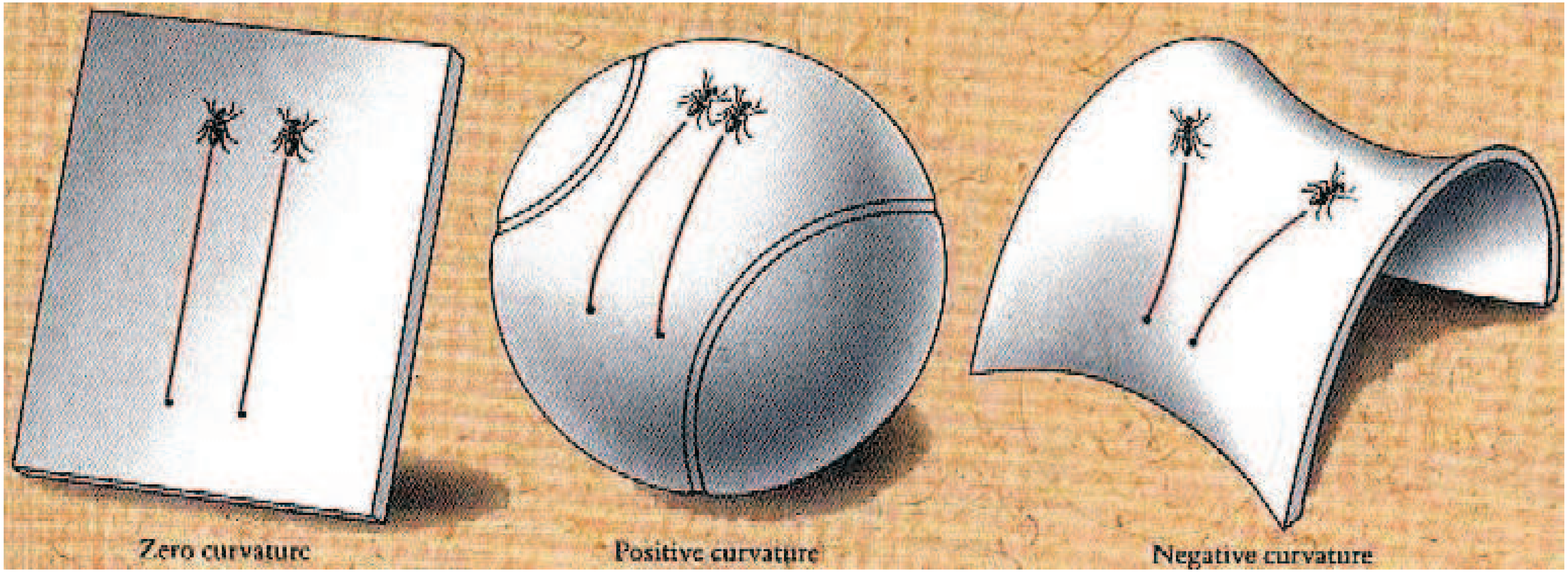
Effect of the curvature of the ambient space?

Euclidean space $\mathbb{R}^2$ $\longrightarrow$ Riemannian manifold



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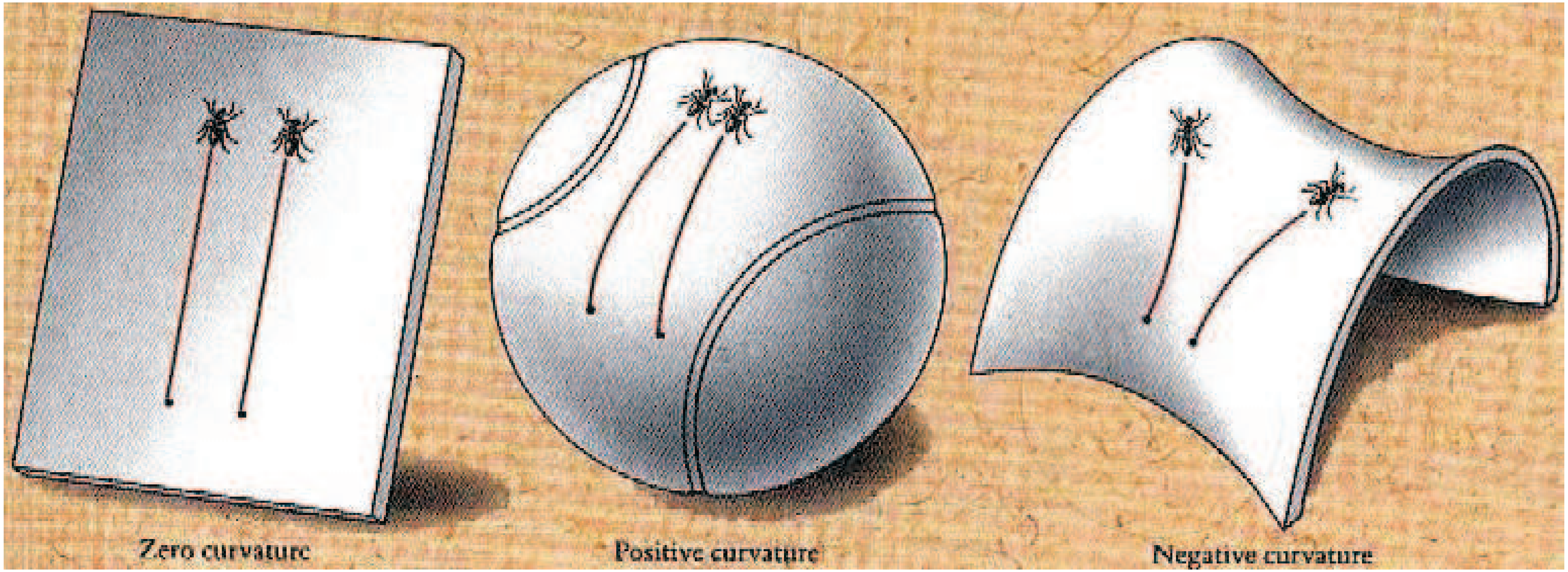

critical | bad | good

Our stochastic model

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$$\frac{\partial p}{\partial \tau} - \Delta p = 0$$

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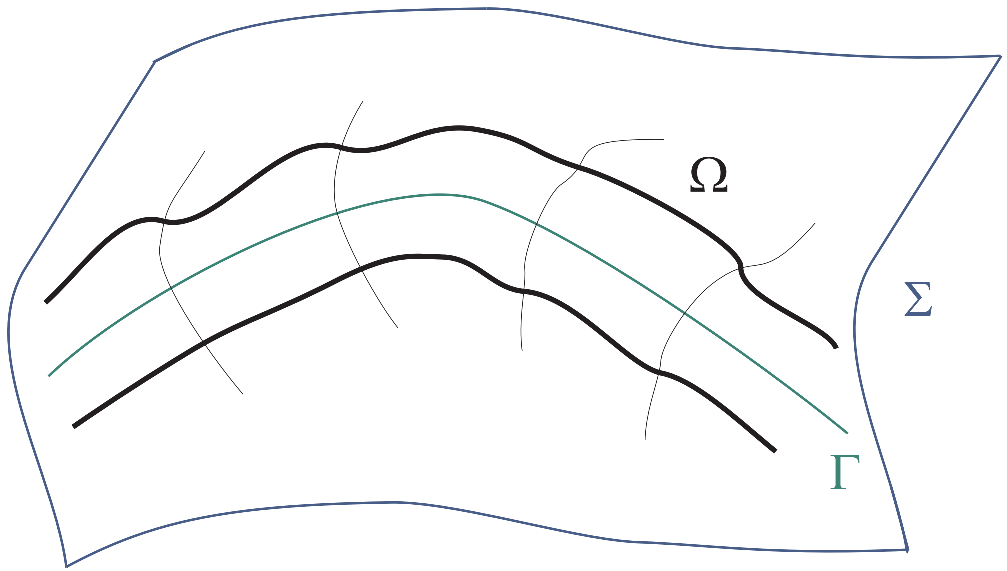
i.e. surface Σ of curvature K



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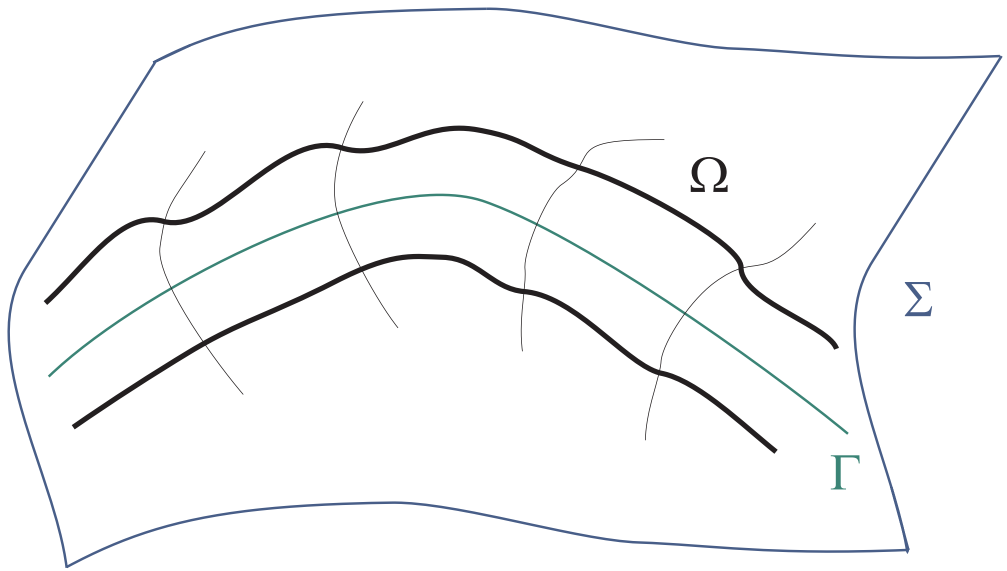

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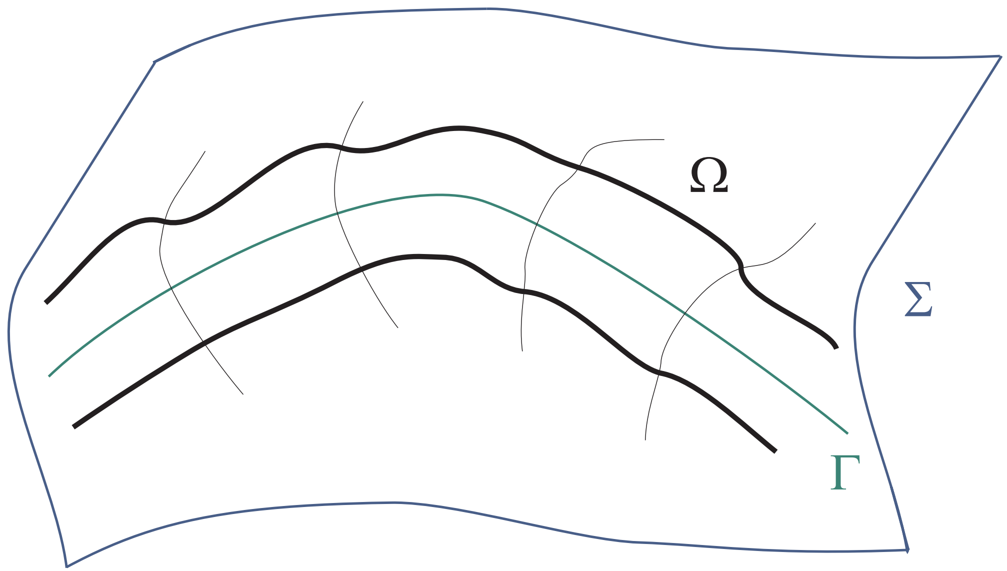

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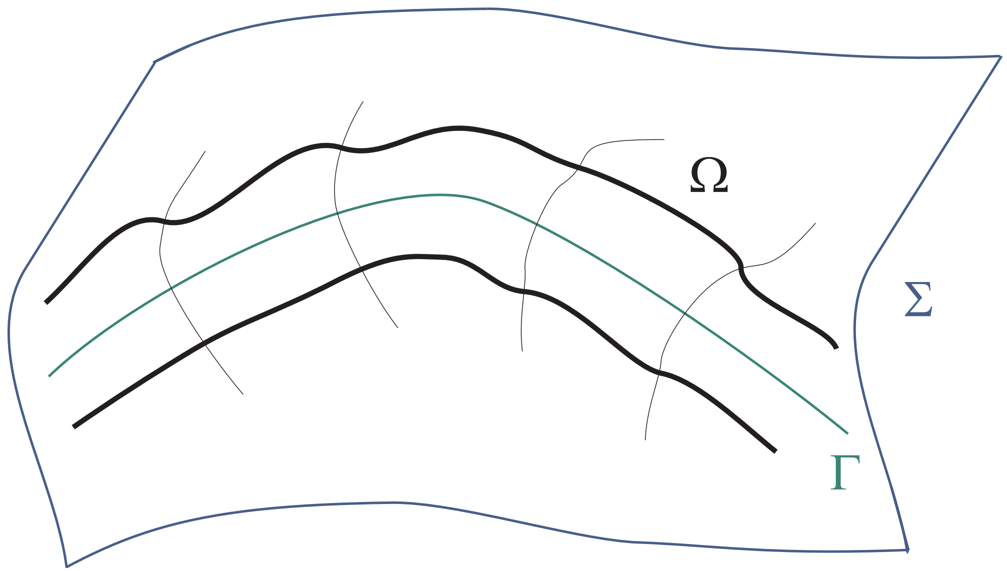

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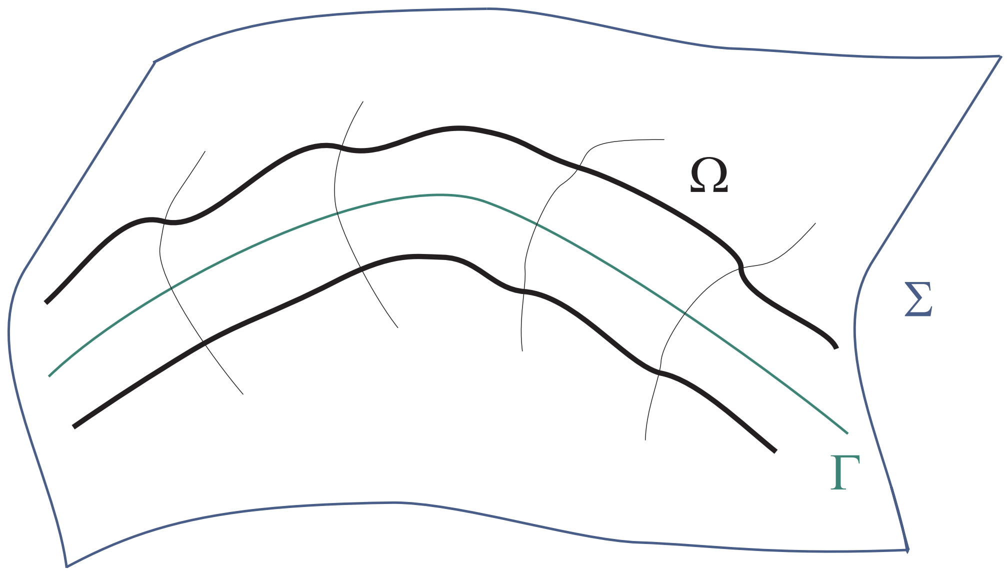

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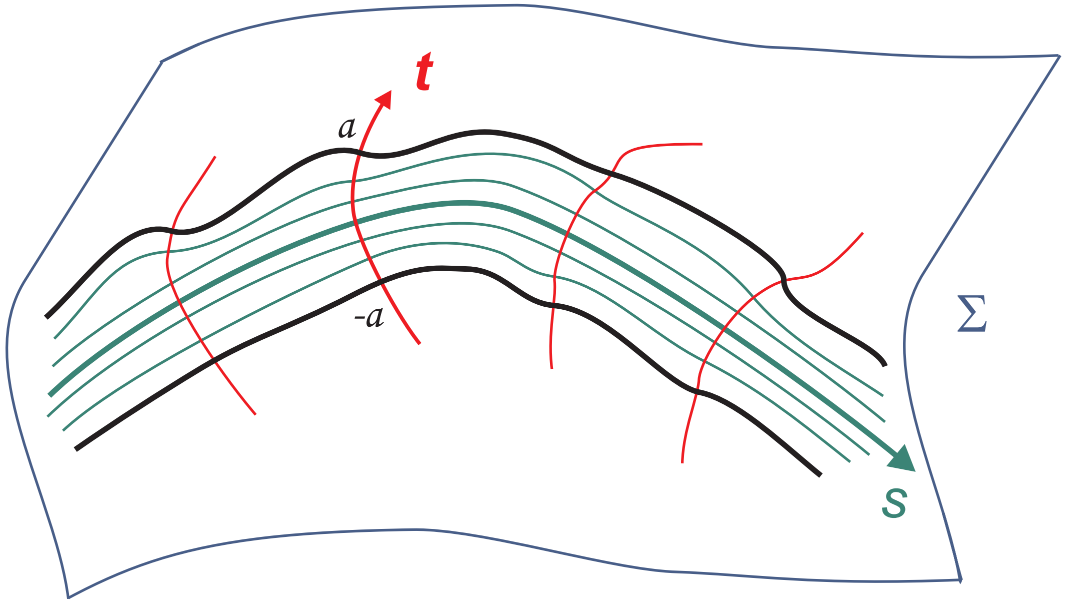
¿ $K \longleftrightarrow$ large-time behaviour of $p(x, \tau)$?

Fermi coordinates

Enrico Fermi (1901–1954)



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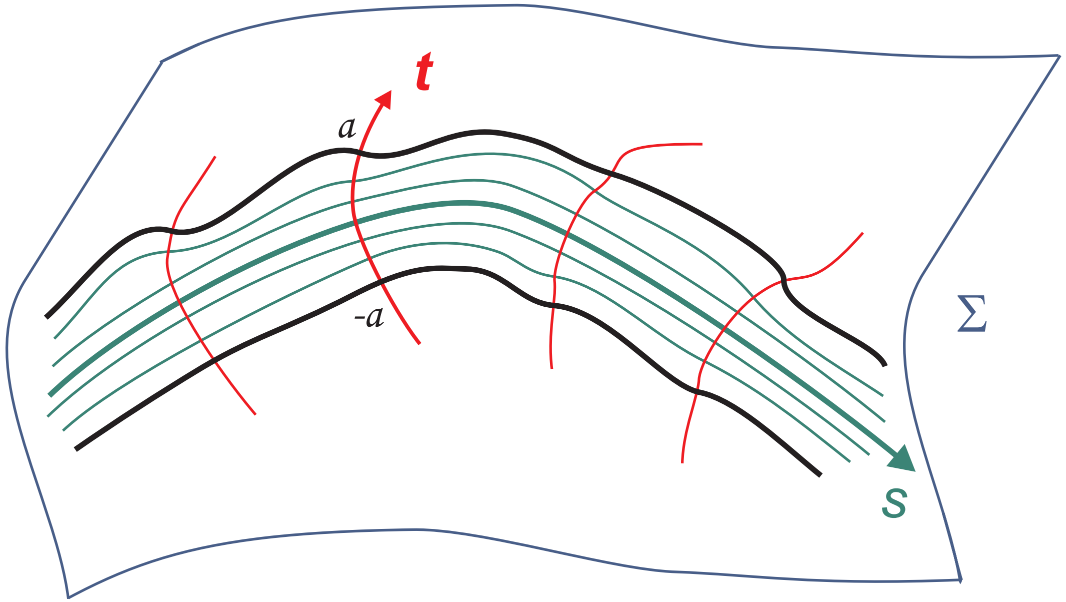

normal

$$\boxed{\Omega := \mathcal{L}(\Omega_0)} \qquad \Omega_0 := \mathbb{R} \times (-a, a), \quad \mathcal{L} : \mathbb{R}^2 \to \Sigma : \quad \mathcal{L}(s,t) := \exp_{\Gamma(s)} \big(t\, N(s)\big)$$

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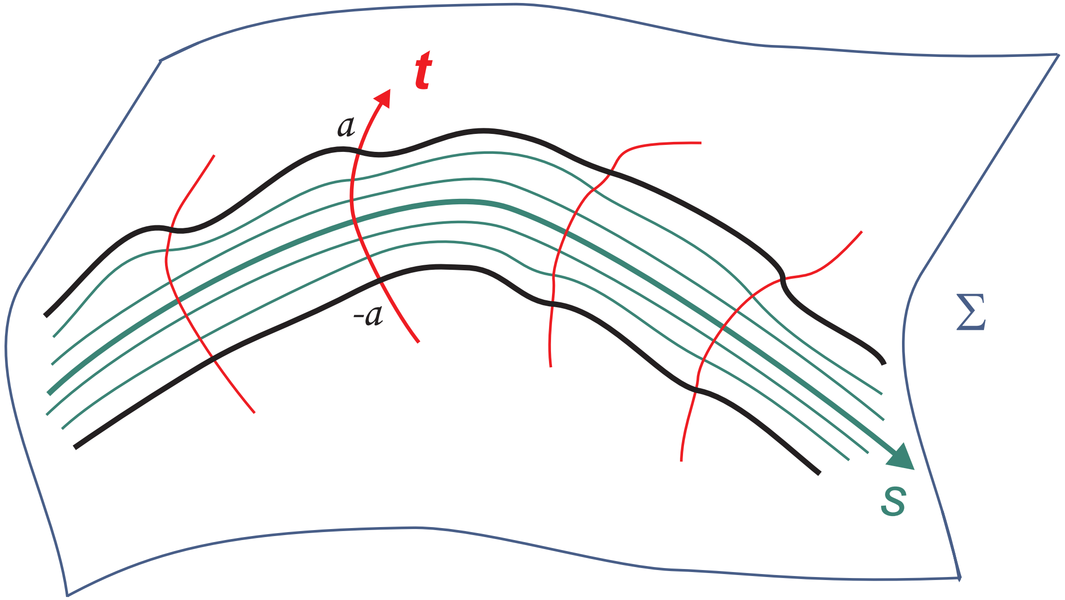

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Jacobi equation

Gauss lemma $\Rightarrow$ metric $G = h(s,t)^2 ds^2 + dt^2$

$$\begin{cases} \boxed{h_{,tt}(s,t) + K(s,t)\, h(s,t) = 0} \\ h(s,0) = 1 \qquad h_{,t}(s,0) = -k \end{cases}$$

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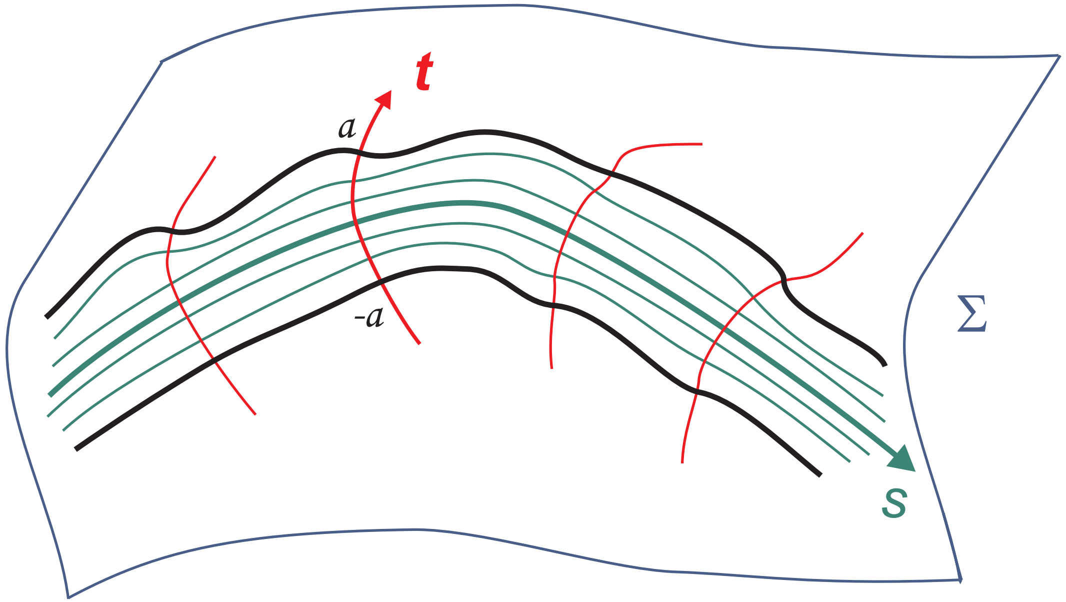

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Laplace-Beltrami operator $\boxed{-\Delta = -|G|^{-\frac{1}{2}} \partial_i |G|^{\frac{1}{2}} G^{ij} \partial_j}$ on $L^2\big(\Omega_0, |G(s,t)|^{\frac{1}{2}} ds\, dt\big)$

Quasi-1-dimensional traveller

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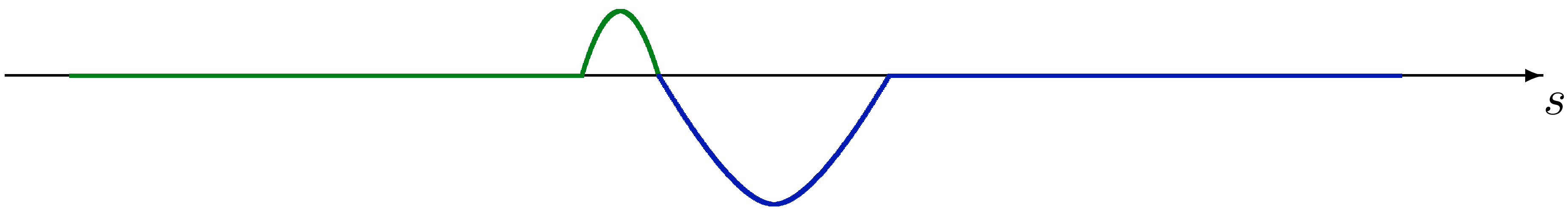
!!! geometrically induced "quantum" $\begin{cases} \text{well} & \Leftarrow \; K \geq 0 \quad \vee \quad k \neq 0 \\ \text{barrier} & \Leftarrow \; K \leq 0 \quad \wedge \quad k = 0 \end{cases}$



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$$\begin{aligned} h^{\frac12}(-\Delta)h^{-\frac12} &= -|G|^{-\frac14}\partial_i |G|^{\frac12} G^{ij}\partial_j |G|^{-\frac14} \qquad \text{on} \qquad L^2\big(\mathbb{R}\times(-a,a), ds\,dt\big) \\ &= -\partial_i G^{ij}\partial_j + V \\ &= -\partial_s\, h(s,t)^{-2}\,\partial_s - \partial_t^2 - \frac54\frac{(h_{,s})^2}{h^4} + \frac12\frac{h_{,ss}}{h^3} - \frac14\frac{(h_{,t})^2}{h^2} + \frac12\frac{h_{,tt}}{h} \\ &= \quad -\partial_s^2 - \partial_t^2 - \tfrac12 K(s,0) - \tfrac14 k(s)^2 + \mathcal{O}(t) \end{aligned}$$

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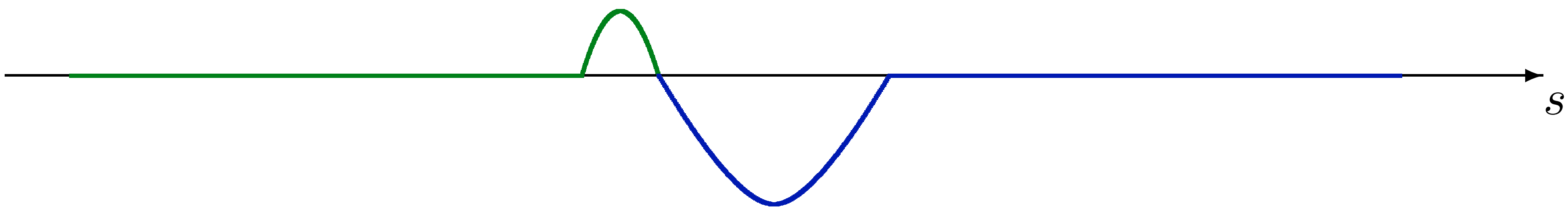
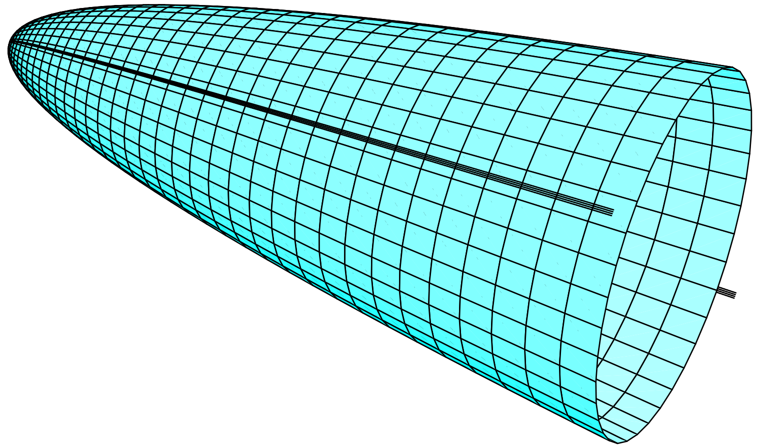


- Heuristics: [Mitchell 2001] • Rigorous treatment: [Freitas, D.K. 2008], [Wittich 2008]
- Abstract approach: [D.K., Raymond, Royer, Siegl 2017]

Parabolic traveller

$K \geq 0$



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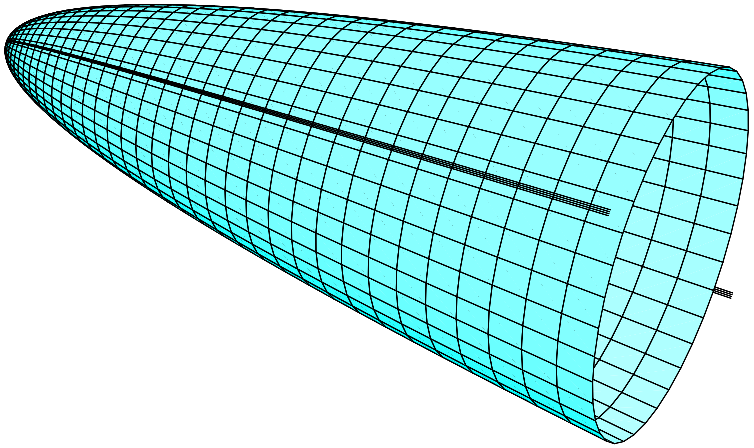

Theorem ([D.K. 2003 (JGP)]).

If K vanishes at infinity then $\sigma_{\text{ess}}(-\Delta) = [E_1, \infty)$

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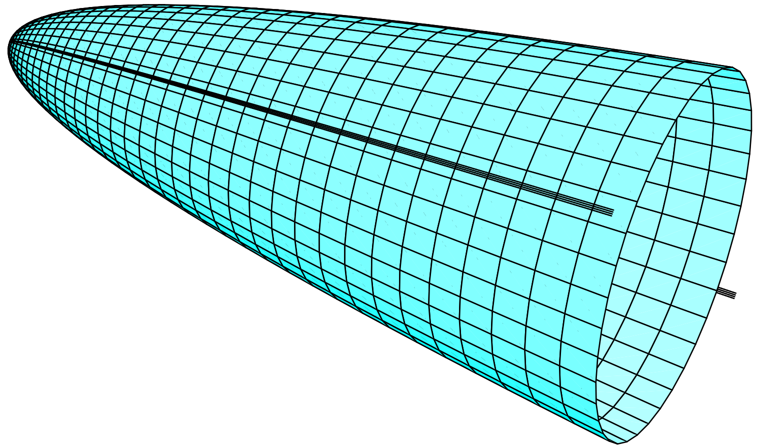

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Corollary. $\sigma_{\text{disc}}(-\Delta) \neq \varnothing$

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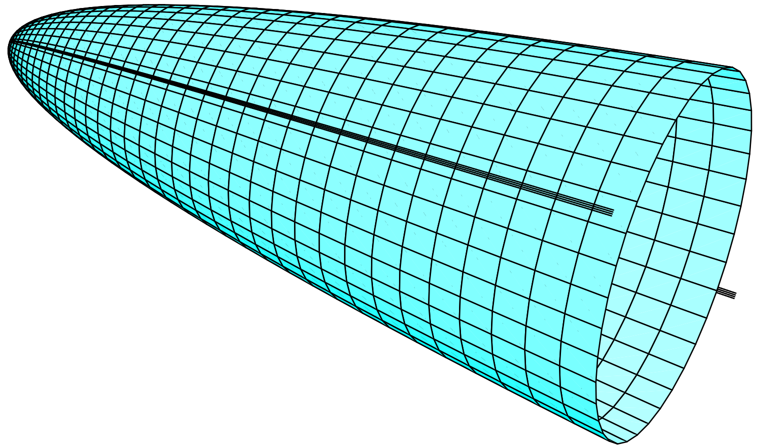

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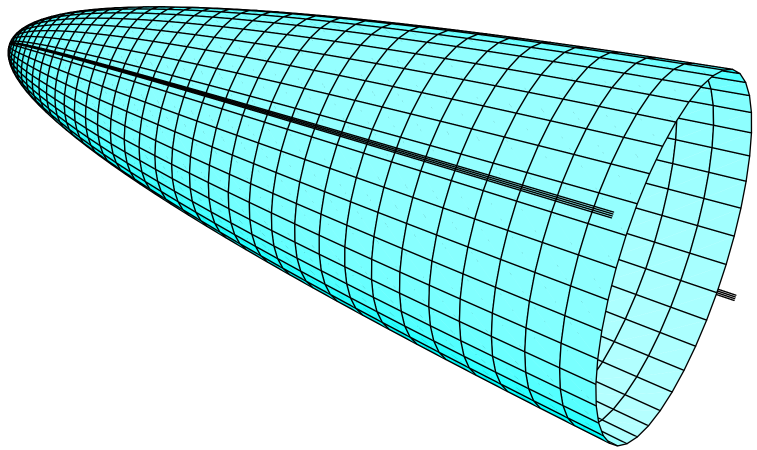

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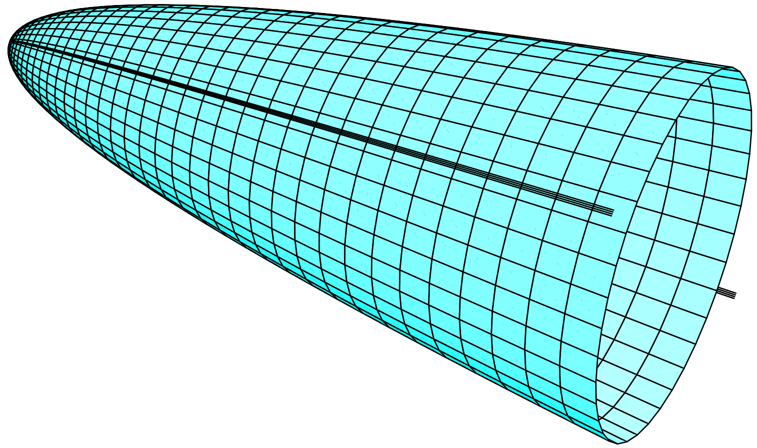

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Proof. Variational: $\exists \psi$ such that $\langle \psi, (-\Delta - E_1)\psi \rangle < 0.$ *q.e.d.*

Hyperbolic traveller

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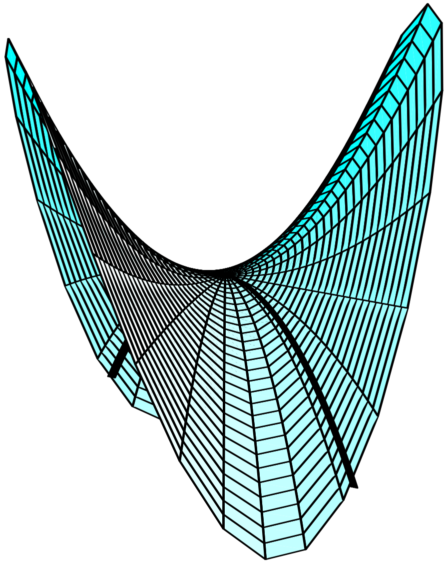

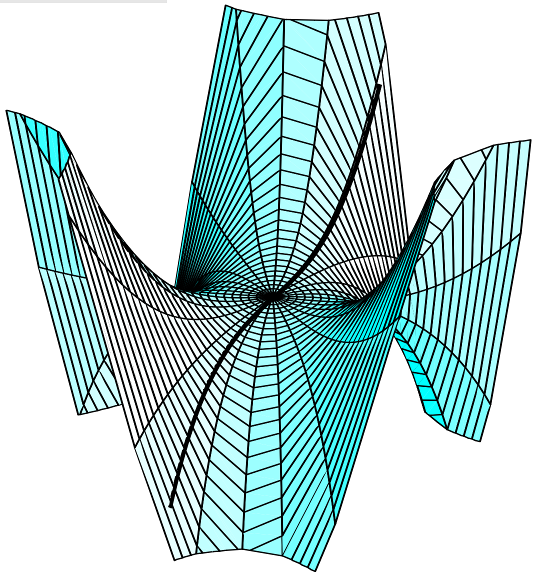

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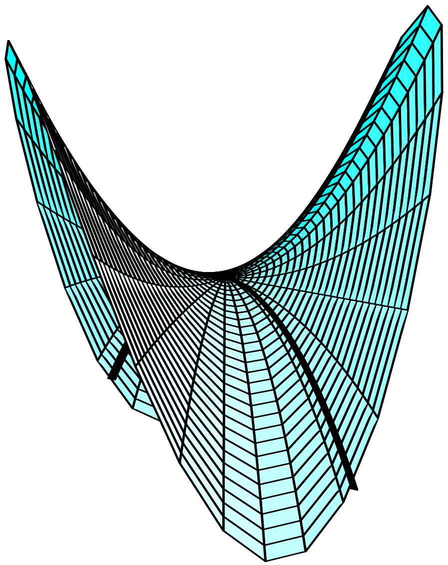

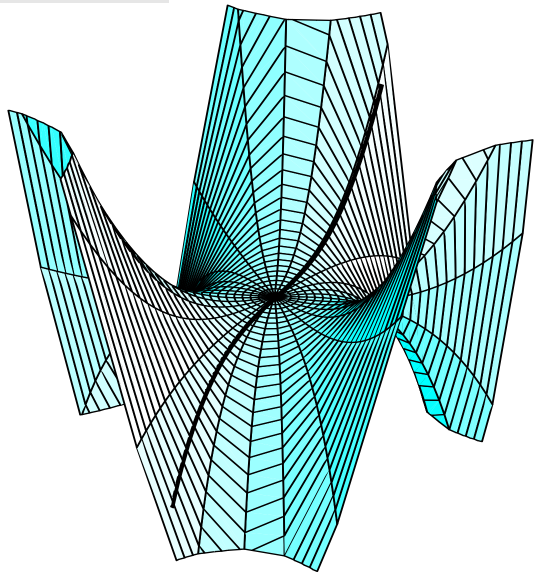

Hyperbolic traveller

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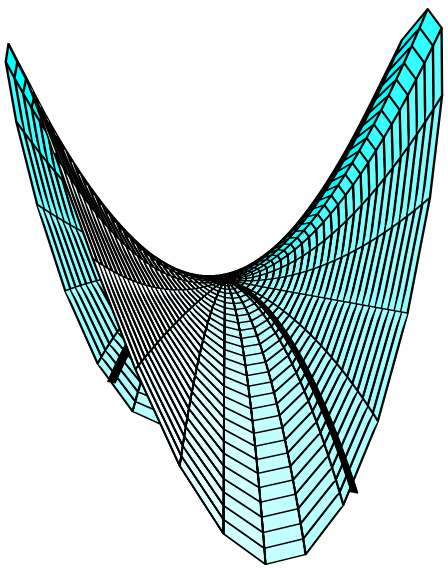

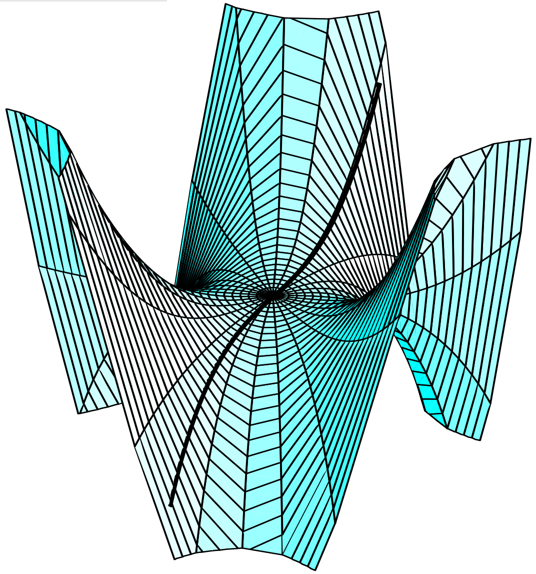

Theorem ([D.K. 2006 (JIA)], [Kolb, D.K. 2014 (JST)]).

If $K \leq 0$ and $k = 0$ and $a \ll 1$ then $$-\Delta - E_1 \geq \frac{c}{1+s^2}$$ Hardy inequality

where $c > 0$ if K is not identically zero and has compact support.

Corollary.

& spectral stability (subcriticality)

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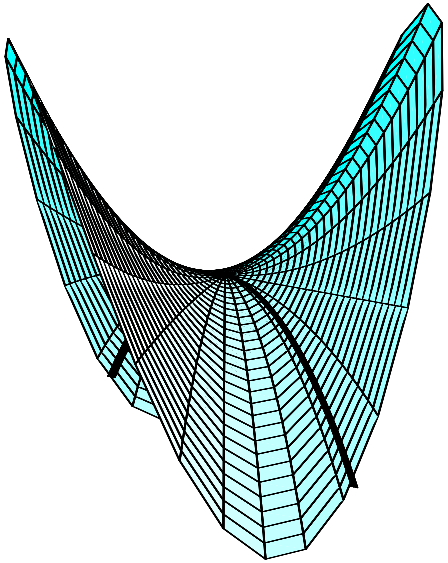

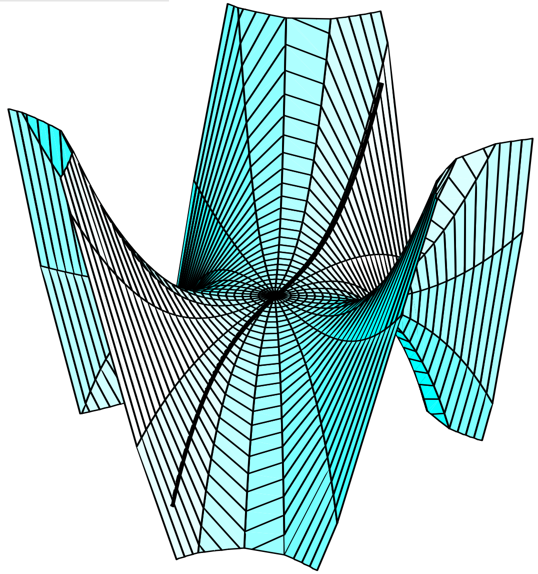

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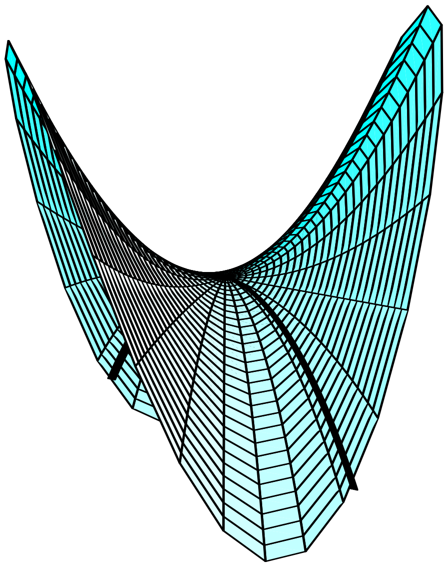

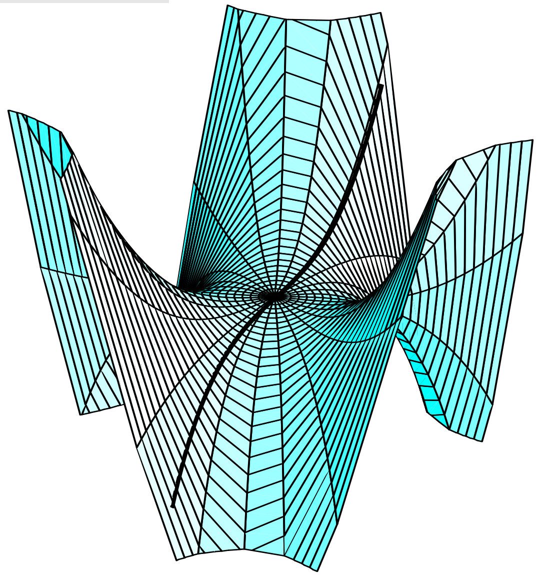

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Proof. Self-similarity transform + weighted Sobolev spaces + Hardy inequality *q.e.d.*

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Model : quasi-1-dimensional Brownian particle in a 2-dimensional curved space

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Moral :

curvature	positive	zero	negative
transport	*bad*	*critical*	*good*
probability decay	$e^{(E_1-\lambda_1)\,\tau}\,e^{-E_1\tau}$	$\tau^{-1/4}\,e^{-E_1\tau}$	$\tau^{-3/4}\,e^{-E_1\tau}$ *

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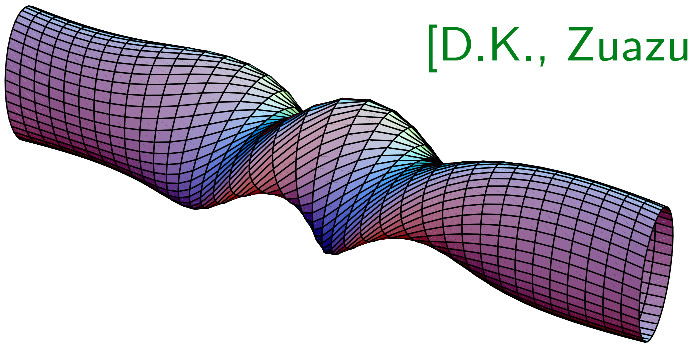
Analogy : (negative curvature)

→ **twisting**



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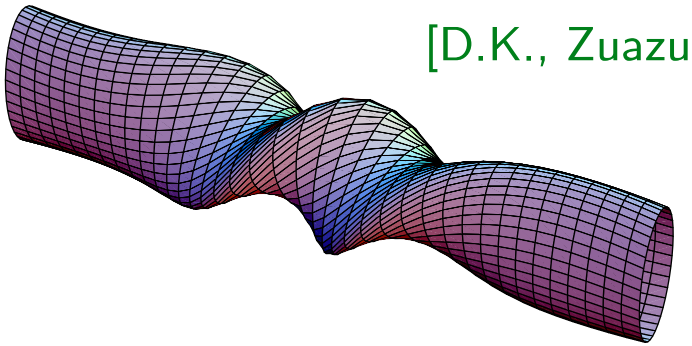

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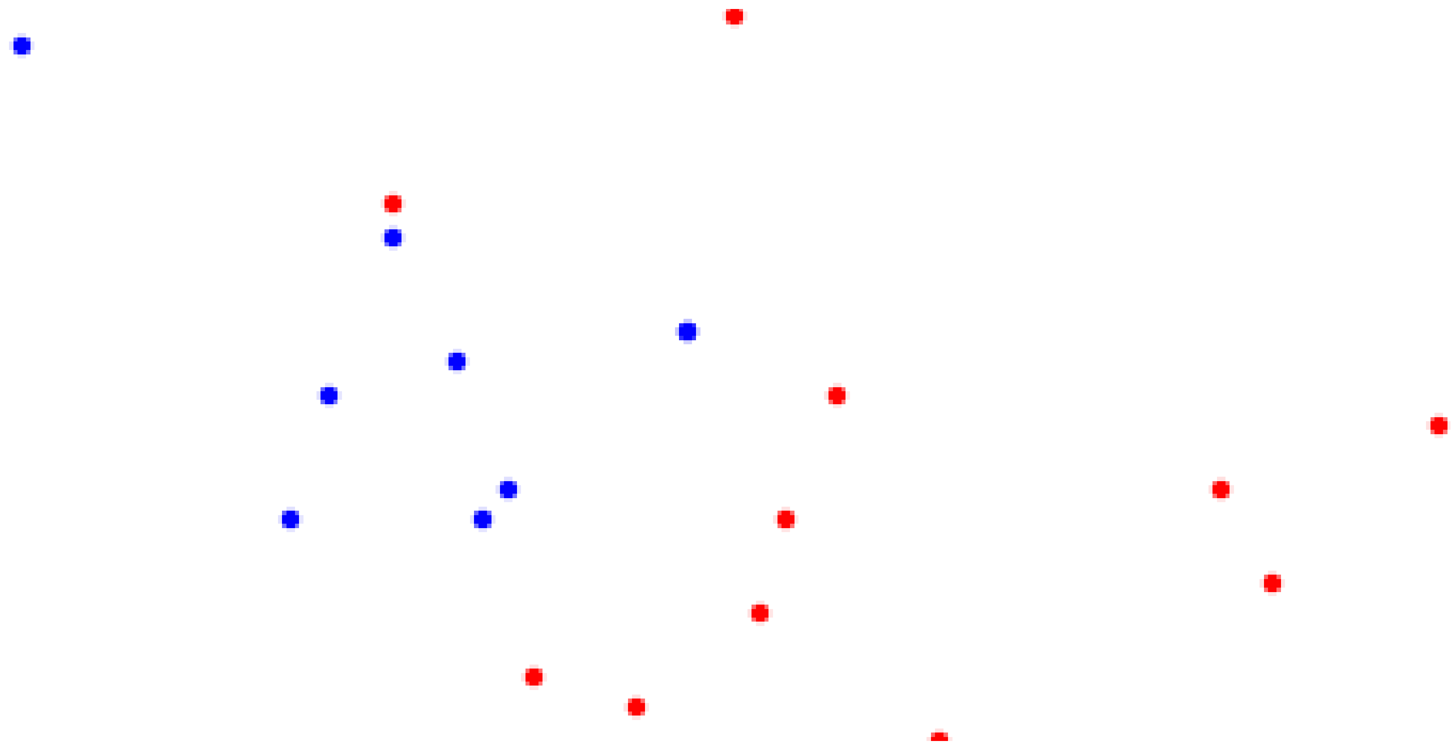
Open problems:

¿ better topology than $L^2_w \to L^2$ with $w(x) = e^{x^2/4}$?

¿ slow decay of curvature at infinity ?

¿ general conjecture: Hardy inequality ⇒ faster cool down ?

Happy birthday, Petr!



Yigal Ozeri: *Territory*, 2012 (oil on canvas, 80 x 120 in)