Quantum SU(3) systems: (non)integrability, quantum chaos, classical limit(s), and experiments

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Chaos, and what it can reveal, Hradec Králové

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Happy birthday, Petr!

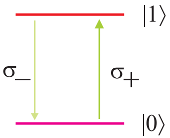
Here should be a photo, but I do not have any...

‘Quantum chaology’

- What is quantum chaos?
- No easy answer, A gimmick: ‘quantum chaology’- investigations of quantum properties of systems that are classically chaotic.
- Which properties?
- What does it mean that a quantum system is classically chaotic/nonitegrable? (Some concrete, and fairly universal notion of ‘classical limit’ needed).
- How to check that the system is indeed chaotic/nonintegrable in this limit?

(Semi)-classics of systems of two-level atoms

- A two-level system



- N atoms

$$S_i := \sum_{n=1}^{N} \sigma_i^{(n)}, \quad i = +, -, z$$

- S_i fulfill the same commutation relations as $\sigma_-, \sigma_+, \sigma_z$
- more technically: they span some of the same Lie algebra as the single-atom operators do (the algebra $\mathfrak{sl}_2(\mathbb{C})$) but of a different dimension (reducible - 'addition of spins')
- (Semi)-classics: $N \to \infty$

Quantum evolution, expectation values, classical evolution

- (Effective) Hamiltonian: $H = H(\hbar \mathbf{S})$
- H commutes with all Casimir invariants of the relevant real algebra ($\mathfrak{su}_2, \mathfrak{su}_3$) - we can treat independently each irreducible representation
- Heisenberg equations of motion: $\frac{d}{dt} S_i = \frac{i}{\hbar}[H, S_i] = F_i(\mathbf{S})$
- Expectation values: $s_i := \hbar \langle S_i \rangle := \hbar \langle \psi | S_i | \psi \rangle$
- Dynamics: $\frac{d}{dt} s_i = \hbar \langle F_i(\mathbf{S}) \rangle \neq \hbar F_i(\langle \mathbf{S} \rangle)$
- In the classical limit "$\hbar \to 0$" (which should correspond roughly to $N \to \infty$ or the dimension of the representation going to infinity = "large quantum numbers"), we would like to have $\frac{d}{dt} s_i = f_i(\mathbf{s})$, for appropriately scaled s_i (with N or the dimension of a representation).
- Here, the scaling: $s_i = \langle S_i \rangle / N$ (obvious?); the dimension of the largest irreducible representation is proportional to N so both scalings coincide.

More technically...

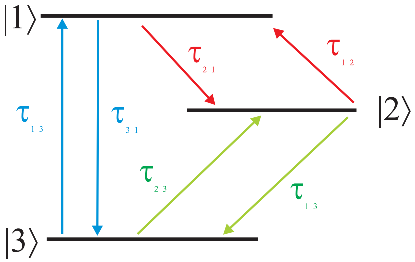
- **Problem:** Find an appropriate classical phase-space (manifold with a canonical=Poisson structure) for the classical evolution.
- We look for a classical phase space M equipped with Poisson brackets $\{\,,\}$ and a map $\tilde{\mu}$ from the algebra of observables (here $\mathfrak{sl}(2\mathbb{C})$, such that (Dirac quantisation condition),

$$\tilde{\mu}([a,b]) = \{\tilde{\mu}(a), \tilde{\mu}(b)\}$$

- **General procedure**
  - M is the manifold of coherent states (all states obtained from the unique eigenvector of S_z with the largest spin component along z bu unitary transformations of the corresponding group $SU(2)$).
  - M is two-dimensional, (in fact it is the unit sphere)
  - On M there is a natural Poisson structure,
  - $\tilde{\mu}(a)(\gamma) = \langle\gamma|a|\gamma\rangle$ fulfills Dirac q. c. (γ is a point on the sphere).
- **Classical limit** = dimension of representation $\to \infty$ ("large spin")
- **Remark** For two-level systems we can't expect anything interesting: the phase space is two-dimensional (one degree of freedom), the Hamiltonian itself is a constant of motion = each system is integrable.

Multilevel systems

- Three levels



- N-atoms: $S_{ij} := \sum_{n=1}^{N} \tau_{ij}^{(n)}$
- Applications
  - Semiclassical laser theory (Seeger, Kolobov, Kuś, Haake, 1996)
  - Nuclear shell models
  - Cold atom/ion systems (Graß, Juliá-Díaz, Kuś, Lewenstein, 2013)
  - ...

- The whole construction works (*mutatis mutandis*) also in this case
  - Classical phase space M - manifold of coherent states with the unique, natural Poisson brackets
  - Classical dynamics: $\frac{d}{dt}s = \{H_{cl}, s\}$
    s, H_{cl} - appropriately scaled expectation values in coherent states.
  - The number of classical degrees of freedom: $f = \dim M/2$
  - The proper scaling: $s = \hbar\langle S\rangle$, $\hbar \to 0$
  - The volume of the phase space $\sim \hbar^{-f}$
- For two-level systems the situation is simple, $\hbar \sim 1/j = N/2$ - the total spin

  There is only one way to the classical limit $j \to \infty$

Two vs three (and more) levels

- For three-level systems it is more interesting
  - A representation and, consequently, its dimension and the coherent states are uniquely determined not by one number (j in the two-level case), but bu two such numbers (say λ_1 and λ_2) in fact by the maximal eigenvalues of the common eigenvector of two commuting operators

$$T_3 := (S_{11} - S_{22}) \, , \quad Y := (S_{11} + S_{22} - 2S_{33})$$

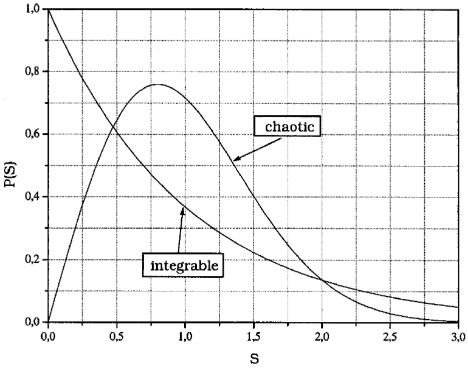
  - The classical limit (corresponding to $j \to \infty$ for two-level systems) is $(\lambda_1, \lambda_2) = k(c_1, c_2)$, $k \to \infty$, c_1, c_2 - fixed (Schäfer, Kuś 2006, 2007)
    - a 6 - dimensional classical phase-space (3 degrees of freedom) for $\lambda_1 \lambda_2 \neq 0$
    - a 4 - dimensional classical phase-space (2 degrees of freedom) for vanishing λ_1 or λ_2
  - Hamiltonians for systems with 2 degrees of freedom with an additional integral of motion are integrable (no chaos)
  - Hamiltonians for systems with 3 degrees of freedom are (in general) chaotic, if there is only one additional constant of motion
  - Quantum Hamiltonian (Gnutzmann, Haake, Kuś, 2000)

$$H = BT_3 + J\hbar^2 \left(S_{12}^2 + S_{21}^2 \right) + K\hbar^2 \left(S_{13}S_{32} + S_{23}S_{31} \right)$$

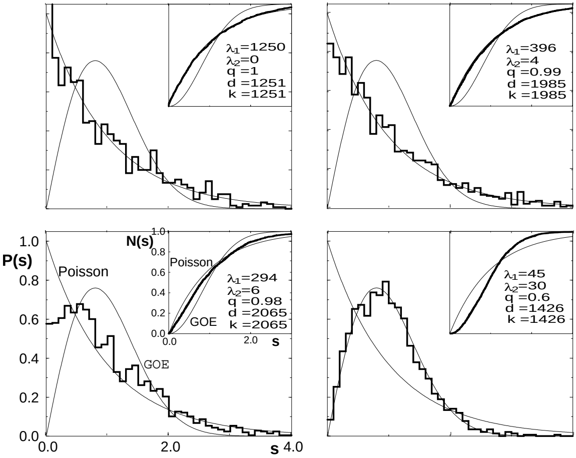
    has and additional constant of the motion $[H, Y] = 0$
  - Depending on how we reach the classical limit representations we obtain integrable (non-chaotic) and (possibly) nonintegrable (chaotic) classical system

Quantum signatures of chaos

Classical integrability can be traced on the quantum level by distribution of spacing between neighboring energy levels



Level spacings



Gnutzmann, Haake, Kuś, 2000.

How to prove that a system is non-integrable?

- A general idea: if a nonlinear system of differential equations is integrable then its linearization is also integrable
- Hence: if the linearization is non-integrable so is the original system
- Hence: the problem reduces to showing that the linearization

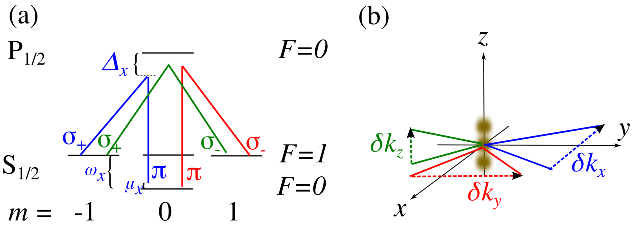
$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$$

  where a_k are some functions of the independent variable (time) is non-integrable
- One mimics then the classical Galois theory concerning solubility of **algebraic** (polynomial) equations
  - extend the field of coefficients by new elements obtained by algebraic operations and taking roots until the filed contains all solutions
  - determine the Galois group = the group of all automorphisms of the extended field leaving the original field of coefficients invariant
  - if the group is solvable (a technical notion from the group theory) the algebraic equation can be solved by algebraic means.
- For differential equations one proceeds analogously
  - extends the field of coefficients a_k by taking algebraic functions of them, integrals of them and integrals of exponentials of them ('integrability in quadratures' equivalent by the Liouville-Arnold theorem to existence of needed number of integrals of motion)
  - if the corresponding Galois group is not solvable the equation in not integrable in quadratures = there in not enough integrals of motion

(Sawicki, Kuś, 2010)

How to do it (experimentally)?

- Ions (Yb^+) in a linear trap + three pairs of counter-propagating laser beams creating standing waves



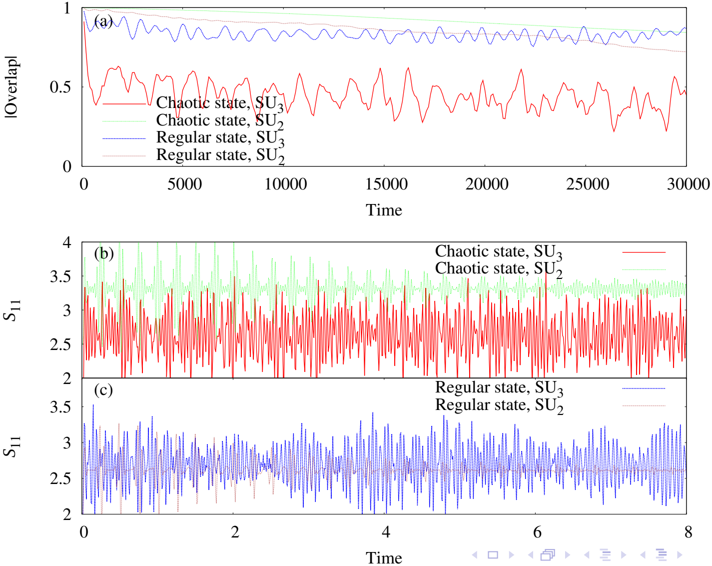
- Ions are in the trapping potential and coupled by the Coulomb repulsion, phonon modes couple to spin operators S_{kl} (the interaction of ions with standing waves is position-dependent)
- Effective Hamiltonian for spin variables

$$H = BT_3 + J\sum_{k<l} S_{kl}S_{lk}$$

has the desired properties
(T. Graß, B. Juliá-Díaz, M. Kuś, M. Lewenstein, Phys. Rev. Lett. **111**, 090404 (2013))

Dynamics

- It is hard to recover the spectrum
- The number of ions is (at most) moderate
- Signature of quantum chaos - sensitivity to parameter changes



Outlook. Work in progress

- Integrability in the classical limit (dimension of representation $\to \infty$) visible already in quantum case for large dimensions of representations
- This fact should be recoverable from analysis of the relevant differential Galois groups
- In the coherent states representation

$$S_{ij} = a_{ij}(\gamma; \lambda_1, \lambda_2) + \sum_k b_{ij}^k(\gamma; \lambda_1, \lambda_2) \frac{\partial}{\partial \gamma_i}$$

  where $\gamma_i \in \mathbb{C}$ are coordinates on M for given (finite) λ_1, λ_2
- All relevant operators (in particular, the Hamiltonian) are expressible *via* above...
- The differential Galois group of the dynamical equations depends on (λ_1, λ_2)
- How to find it (in principle we know, what should be done, but...

Summary

- "Exotic" quantum chaotic system
- Experiment accessible (or at lest, close to...) using available technologies
- Differences between chaotic and regular system visible "far from classical limit"
- History
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  - K. Karnas, M. Kuś *in progress*