Fermi's rule and high-energy asymptotics for quantum graphs

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Description of the model

- set of ordinary differential equations
- graph consists of set of vertices $\mathcal{V}$, set of not oriented edges (both finite $\mathcal{E}$ and infinite $\mathcal{E}_\infty$).
- Hilbert space of the problem

$$\mathcal{H} = \bigoplus_{(j,n)\in I_\mathcal{L}} L^2([0, l_{jn}]) \oplus \bigoplus_{j\in I_\mathcal{C}} L^2([0,\infty))\,.$$

- states described by columns

$$\psi = (f_{jn} : \mathcal{E}_{jn} \in \mathcal{E},\, f_{j\infty} : \mathcal{E}_{j\infty} \in \mathcal{E}_\infty)^T.$$

- the Hamiltonian acting as $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ – corresponds to the Hamiltonian of a quantum particle for the choice $\hbar = 1$, $m = 1/2$

Domain of the Hamiltonian

- domain consisting of functions in $W^{2,2}(\Gamma)$ satisfying coupling conditions at each vertex
- coupling conditions given by

$$(U_v - I_v)\Psi_v + i(U_v + I_v)\Psi'_v = 0\,.$$

where $\Psi_v = (\psi_1(0), \ldots, \psi_d(0))^{\mathrm{T}}$ and $\Psi'_v = (\psi_1(0)', \ldots, \psi_d(0)')^{\mathrm{T}}$ are the vectors of limits of functional values and outgoing derivatives where d is the number edges emanating from the vertex v and U_v is a unitary $d \times d$ matrix

Examples of coupling conditions

- δ-**conditions**

$$\begin{aligned} f(\mathcal{X}) &\equiv f_i(\mathcal{X}) = f_j(\mathcal{X}) \quad \text{for all } i,j \in \{1,\dots,n+m\} \\ \sum_{j=1}^{n+m} f_j'(\mathcal{X}) &= \alpha f(\mathcal{X}) \end{aligned}$$

- δ'_{s}-**conditions**

$$\begin{aligned} f'(\mathcal{X}) &\equiv f_i'(\mathcal{X}) = f_j'(\mathcal{X})\,, \quad \text{for all } i,j \in \{1,\dots,n+m\} \\ \sum_{j=1}^{n+m} f_j(\mathcal{X}) &= \beta f'(\mathcal{X})\,. \end{aligned}$$

- **standard conditions** (sometimes called Kirchhoff) represent a special case of δ-condition for $\alpha = 0$.
- **Dirichlet conditions** mean that all the functional values are zero at the vertex.
- **Neumann conditions**, on the other hand, mean that all the derivatives vanish at the vertex.

Resolvent resonances

- poles of the meromorphic continuation of the resolvent $(H - \lambda \mathrm{id})^{-1}$
- another definition: $\lambda = k^2$ is a resolvent resonance if there exists a generalized eigenfunction $f \in L^2_{\mathrm{loc}}(\Gamma)$, $f \not\equiv 0$ satisfying $-f''(x) = k^2 f(x)$ on all edges of the graph and fulfilling the coupling conditions, which on all external edges behaves as $c_j \, \mathrm{e}^{ikx}$.

Fermi's rule for graphs with standard condition

Theorem (Lee, Zworski)

Consider a simple eigenvalue $k_0^2 > 0$ of the Hamiltonian $H \equiv H(0)$ and let u be the corresponding eigenfunction. Then for $|k| \leq k_{\max}$ there exists a smooth function $t \mapsto k(t)$ such that $k^2(t)$ is the resolvent resonance of $H(t)$. Moreover, we have

$$
\begin{aligned}
\mathrm{Im}\, \ddot{k}(0) &= -\sum_{s=N+1}^{N+M} |F_s|^2\,,\\
F_s &= k_0 \langle \dot{a} u, e^s(k_0)\rangle + \\
&+ \frac{1}{k_0} \sum_{v\in\Gamma} \sum_{e_j \ni v} \frac{1}{4} \dot{a}_j (3 \partial_\nu u_j(v) \overline{e_j^s(k,v)} - u(v) \partial_\nu \overline{e_j^s(k,v)})\,,
\end{aligned}
$$

- double dot denotes the second derivative with respect to t, $\langle \bullet, \bullet \rangle$ is the inner product in $\oplus_{j=1}^{N} L^2([0,\ell_j])) \oplus \oplus_{s=N+1}^{N+M} L^2([0,\infty))$, the sum $\sum_{v\in\Gamma}$ goes through all the vertices of the graph Γ, $\partial_\nu u_j(0) = -u_j'(0)$ and $\partial_\nu u_j(\ell_j) = u_j'(\ell_j)$.
- $\ell_j(t) = \mathrm{e}^{-a_j(t)}\ell_j\,, \quad a_j(0) = 0\,, \quad \dot{a}_j = \dot{a}_j(0)$
- for $k^2 \notin \sigma_{\mathrm{pp}}(H)$ we define generalized eigenfunctions $e^s(k)$, $N+1 \le s \le N+M$ as

$$\begin{aligned} e^s(k) \in \mathcal{D}_{\mathrm{loc}}(H)\,, \quad (H-k^2)e^s(k) = 0\,, \\ e_j^s(k,x) = \delta_{js}\mathrm{e}^{-ikx} + s_{js}(k)\mathrm{e}^{ikx}\,, \quad N+1 \le j \le N+M\,, \end{aligned}$$

where e_j^s are the half-line components of e^s. This family can be holomorphically extended to the points of the spectrum of H and therefore it is defined for all k.

Pseudo orbit expansion for the resonance condition

- there is a known method for finding the spectrum of a compact graph by the pseudo orbit expansion
- the vertex scattering matrix maps the vector of amplitudes of the incoming waves into a vector of amplitudes of the outgoing waves $\vec{\alpha}_v^{\mathrm{out}} = \sigma^{(v)}\vec{\alpha}_v^{\mathrm{in}}$
- for a non-compact graph we similarly define effective vertex scattering matrix $\tilde{\sigma}^{(v)}$

Theorem

Let us assume the vertex connecting n internal and m external edges. The effective vertex-scattering matrix is given by

$$\tilde{\sigma}(k) = -[(1-k)\tilde{U}(k) - (1+k)I_n]^{-1}[(1+k)\tilde{U}(k) - (1-k)I_n]$$

- we define the directed graph Γ_2: each edge of the compact part of Γ is replaced by two directed edges of the same lengths and opposite directions
- **periodic orbit** γ is a closed path on Γ_2
- **pseudo orbit** $\tilde{\gamma}$ is a collection of periodic orbits
- **irreducible pseudo orbit** $\bar{\gamma}$ is a pseudo orbit, which does not use any directed edge more than once
- we define length of a periodic orbit by $\ell_\gamma = \sum_{j, b_j \in \gamma} \ell_j$; the length of pseudo orbit (and hence irreducible pseudo orbit) is the sum of the lengths of the periodic orbits from which it is composed
- we define product of scattering amplitudes for a periodic orbit $\gamma = (b_1, b_2, \dots, b_n)$ as $A_\gamma = S_{b_2 b_1} S_{b_3 b_2} \dots S_{b_1 b_n}$, where $S_{b_2 b_1}$ is the entry of the matrix S in the b_2-th row and b_1-th column; for a pseudo orbit we define $A_{\tilde{\gamma}} = \Pi_{\gamma_n \in \tilde{\gamma}} A_{\gamma_j}$
- by $m_{\tilde{\gamma}}$ we denote the number of periodic orbits in the pseudo orbit $\tilde{\gamma}$

Theorem

The resonance condition is given by the sum over irreducible pseudo orbits

$$\sum_{\bar{\gamma}} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}} \, \mathrm{e}^{ik\ell_{\bar{\gamma}}} = 0 \, .$$

- in general $A_{\bar{\gamma}}$ can be energy dependent, but this is not the case for standard coupling.
- idea of the proof: the permutations in the determinant can be represented as product of disjoint cycles

Fermi's rule for graphs with general coupling

- let the internal graphs edge lengths $\ell_j = \ell_j(t)$ depend on the parameter t as C^2 functions
- suppose that at least some of them are non-constant in the vicinity of $t = 0$ and that at that point the system has an eigenvalue $k_0^2 > 0$ embedded in the continuous spectrum
- $\dot{k} \in \mathbb{R}$, where dot signifies the derivative with respect to t.
- Furthermore, we have

$$\dot{k} \sum_{\bar{\gamma}} \left(\ell_{\bar{\gamma}} A_{\bar{\gamma}}(k) - i \frac{\partial A_{\bar{\gamma}}(k)}{\partial k} \right) (-1)^{m_{\bar{\gamma}}} \, \mathrm{e}^{ik\ell_{\bar{\gamma}}} + \\ + k \sum_{\bar{\gamma}} \dot{\ell}_{\bar{\gamma}} (-1)^{m_{\bar{\gamma}}} A_{\bar{\gamma}}(k) \, \mathrm{e}^{ik\ell_{\bar{\gamma}}} = 0 \,,$$

- we have a (more complicated) condition from which one finds $\ddot{k}$

Example of the trajectory of a resonance

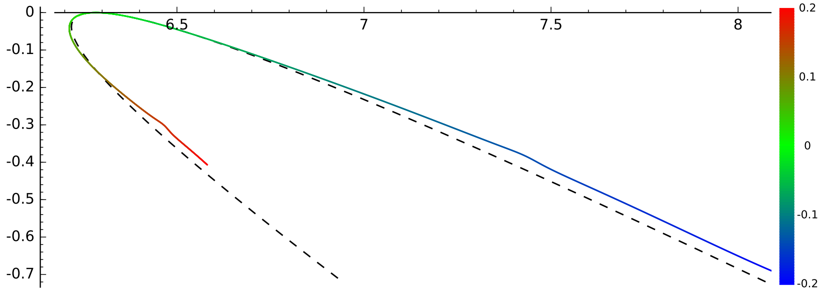

Figure: The resonance trajectory for the graph consisting of a circle with two attached half-lines with δ-conditions coming from the eigenvalue with $k_0 = 2\pi$, $\ell_1 = 1 - t$, $\ell_2 = 1 + 2t$, $\alpha = 10$. The trajectory is shown for $t \in (-0.2, 0.2)$ and it is approximated by the dashed curve $k = k_0 + t\dot{k} + \frac{t^2}{2}\mathrm{Re}\,\ddot{k} + \frac{it^2}{2}\mathrm{Im}\,\ddot{k}$ with $\dot{k} = -\pi$, $\mathrm{Re}\,\ddot{k} = 75.61$, $\mathrm{Im}\,\ddot{k} = -44.41$.

High-energy asymptotics of resonances for δ-coupling

Theorem (Exner, J.L.)

Consider a graph Γ with a δ-coupling at all the vertices. Its resonances converge to the resonances of the same graph with the standard conditions as their real parts tend to infinity.

- idea of the proof: the corresponding vertex scattering matrix for δ-condition converges to the vertex scattering matrix for standard condition

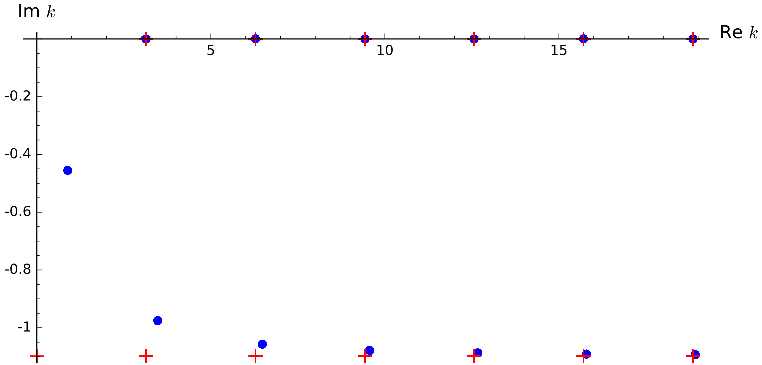

Figure: Illustration to example with a circle and two attached half-lines with δ-conditions with the parameters $\ell_1 = 1$; $\ell_2 = 1$; $\alpha_1 = 1$; $\alpha_2 = 1$. Resonances for δ-condition denoted by blue dots, resonances for standard condition by red crosses.

High-energy asymptotics of resonances for δ'_s-coupling

Theorem (Exner, J.L.)

The resonances of the graph with a δ'_{s} coupling conditions at the vertices converge to the eigenvalues of the graph with Neumann (decoupled) conditions as their real parts tend to infinity.

- idea of the proof: again, the corresponding vertex-scattering matices converge to each other

High-energy asymptotics of resonances for δ'_s-coupling

Theorem (Exner, J.L.)

The resonances of the graph with δ'_s coupling conditions at the vertices, where the half-lines are attached, and arbitrary self-adjoint coupling at the other vertices satisfy

$$\mathrm{Im}\, k \to 0 \quad \text{as} \quad |k| \to \infty\,.$$

Moreover, if the graph is equilateral with δ'_s, then the resonances satisfy

$$\mathrm{Im}\, k_n = \mathcal{O}\left((\mathrm{Re}\, k_n)^{-2}\right)\,, \quad \mathrm{Re}\,(k_n - k_{0n}) = \mathcal{O}\left((\mathrm{Re}\, k_n)^{-1}\right)$$

as $\mathrm{Re}\, k_n \to \infty$, where $k_{0n} = n\pi/\ell_0$.

- idea of the proof: the resonances converge to the eigenvalues of Neumann Hamiltonian, where the half-lines are fully decoupled from the internal part of the graph

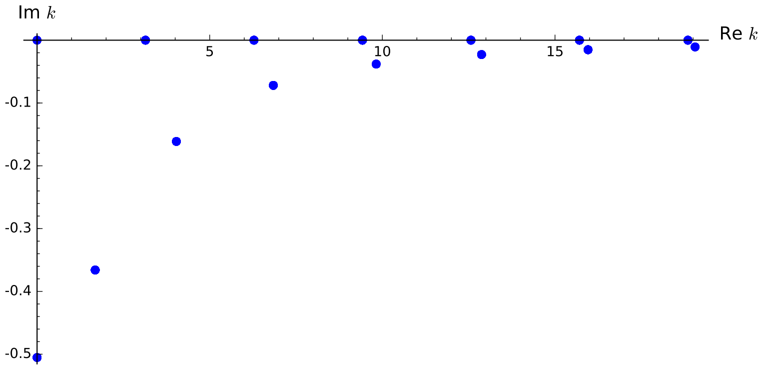

Figure: Illustration to example with a circle and two attached half-lines with δ-conditions with the parameters $\ell_1 = 1$; $\ell_2 = 1$; $\beta_1 = 1$; $\beta_2 = 1$.

Thank you for your attention!

Articles on which the talk was based

M. Lee, M. Zworski: A Fermi golden rule for quantum graphs, *J. Math. Phys.* **57**, 092101 (2016).

P. Exner, J. Lipovský: Pseudo-orbit approach to trajectories of resonances in quantum graphs with general vertex coupling: Fermi rule and high-energy asymptotics, *J. Math. Phys.* **58** (2017), 042101