On the bound states of magnetic Laplacians on wedges

Axel Pérez-Obiol
in collaboration with Pavel Exner and Vladimir Lotoreichik

Department of Theoretical Physics, Nuclear Physics Institute, Řež
Chaos17, Hradec Králové, May 9-11 2017

Outline

Introduction

- Mathematical and physical motivation
- Definition of the mathematical problem and its relation to superconductivty

Existence of bound states

- Neumann and Robin boundary conditions
- Delta interaction on a broken line

Summary and conclusions

Motivation

- **Spectral properties of Schrödinger operators**:
  - Neumann: conjectured that bound states exist on a corner domain for $\phi<\pi$ (proved for $\phi<0.51\pi$)*
  - Delta: discrete spectrum with $B=0$**. Does it persist with $B\neq0$?

*N. Raymond, EMS Tracts in Mathematics, 2017; V. Bonnaillie, Thèse de doctorat, Université Paris XI (2003)

**P. Exner and T. Ichinose, J. Phys. A: Math. Gen. 34 (2001), 1439–1450

Motivation

- **Spectral properties of Schrödinger operators**:
  - Neumann: conjectured that bound states exist on a corner domain for $\phi<\pi$ (proved for $\phi<0.51\pi$)*
  - Delta: discrete spectrum with $B=0$**. Does it persist with $B\neq0$?
- **Quantum mechanics:** is a spinless, massive charged particle on a corner domain of angle ϕ bounded?

*N. Raymond, EMS Tracts in Mathematics, 2017; V. Bonnaillie, Thèse de doctorat, Université Paris XI (2003)

**P. Exner and T. Ichinose, J. Phys. A: Math. Gen. 34 (2001), 1439–1450

Motivation

- **Spectral properties of Schrödinger operators**:
  - Neumann: conjectured that bound states exist on a corner domain for $\phi<\pi$ (proved for $\phi<0.51\pi$)*
  - Delta: discrete spectrum with $B=0$**. Does it persist with $B\neq0$?
- **Quantum mechanics:** is a spinless, massive charged particle on a corner domain of angle ϕ bounded?
- **Superconductivity:** the lowest eigenvalues in the Neumann and Robin conditions limit the critical magnetic field below which there is superconductivity

*N. Raymond, EMS Tracts in Mathematics, 2017; V. Bonnaillie, Thèse de doctorat, Université Paris XI (2003)

**P. Exner and T. Ichinose, J. Phys. A: Math. Gen. 34 (2001), 1439–1450

Magnetic Laplacian on a wedge

- Eigenvalue equation:

$\nabla^2_{\mathbf{A}}\psi = \lambda\psi$, with $\nabla_{\mathbf{A}} := (\mathrm{i}\nabla + \mathbf{A})$, $\mathbf{A}(x_1, x_2) = \frac{1}{2}(-x_2, x_1)^\top$, and

$\psi \in H^1_{\mathbf{A}}(\Omega),\ \ H^1_{\mathbf{A}}(\Omega) := \left\{\psi \in L^2(\Omega)\colon \nabla_{\mathbf{A}}\psi \in L^2(\Omega;\mathbb{C}^2)\right\}$

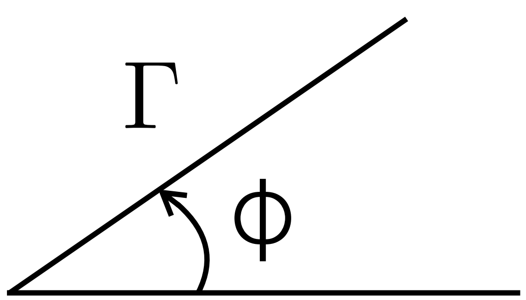
Magnetic Laplacian on a wedge

- Eigenvalue equation:

$\nabla^2_{\mathbf{A}}\psi = \lambda\psi$, with $\nabla_{\mathbf{A}} := (\mathrm{i}\nabla + \mathbf{A})$, $\mathbf{A}(x_1, x_2) = \frac{1}{2}(-x_2, x_1)^\top$, and

$\psi \in H^1_{\mathbf{A}}(\Omega),\ \ H^1_{\mathbf{A}}(\Omega) := \left\{\psi \in L^2(\Omega) \colon \nabla_{\mathbf{A}}\psi \in L^2(\Omega; \mathbb{C}^2)\right\}$

- We consider a wedge of angle ϕ and distinguish three cases:

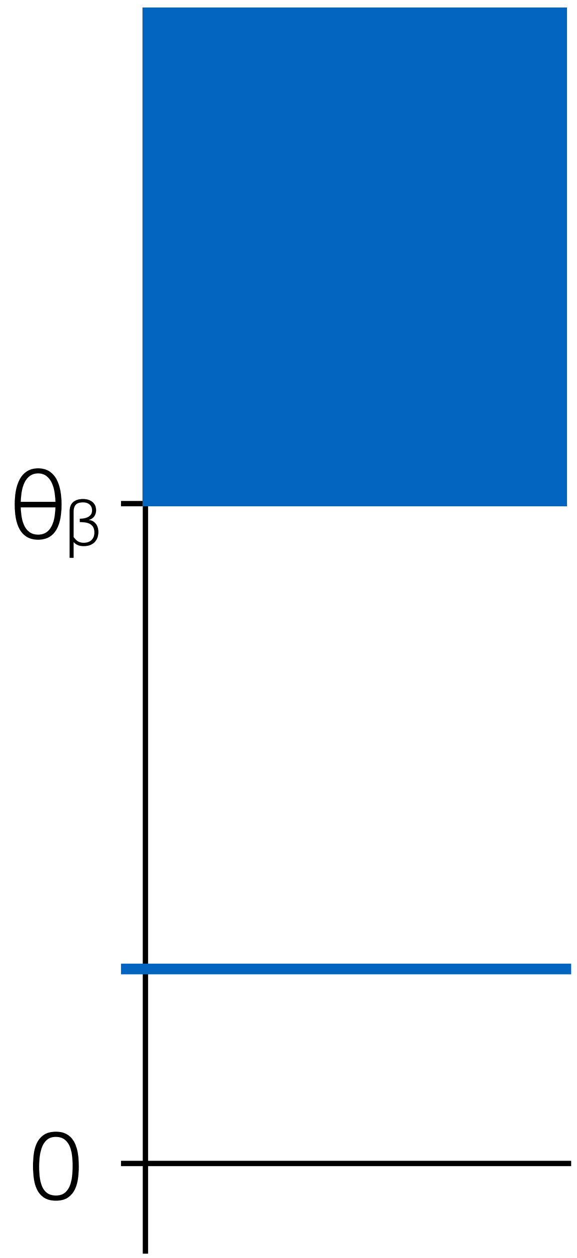


$\Omega_\phi := \left\{(r, \theta) \in \mathbb{R}_+ \times \mathbb{S}^1 \colon\ \in (0, \phi)\right\} \subset \mathbb{R}^2$

- Neumann: $\nabla_{\mathbf{A}}\psi \cdot \hat{n} = 0$, $\Omega := \Omega_\phi$
- Robin: $\nabla_{\mathbf{A}}\psi \cdot \hat{n} = \beta\psi$, $\Omega := \Omega_\phi$
- Attractive δ int. on Γ: $(\nabla_{\mathbf{A}}\psi_+ + \nabla_{\mathbf{A}}\psi_-) \cdot \hat{n} = \beta\psi$, $\Omega := \mathbb{R}^2$

Magnetic Laplacian on a wedge

- Bottoms of essential spectra θ_β found by variational characterisation
- Bound state λ below θ_β?

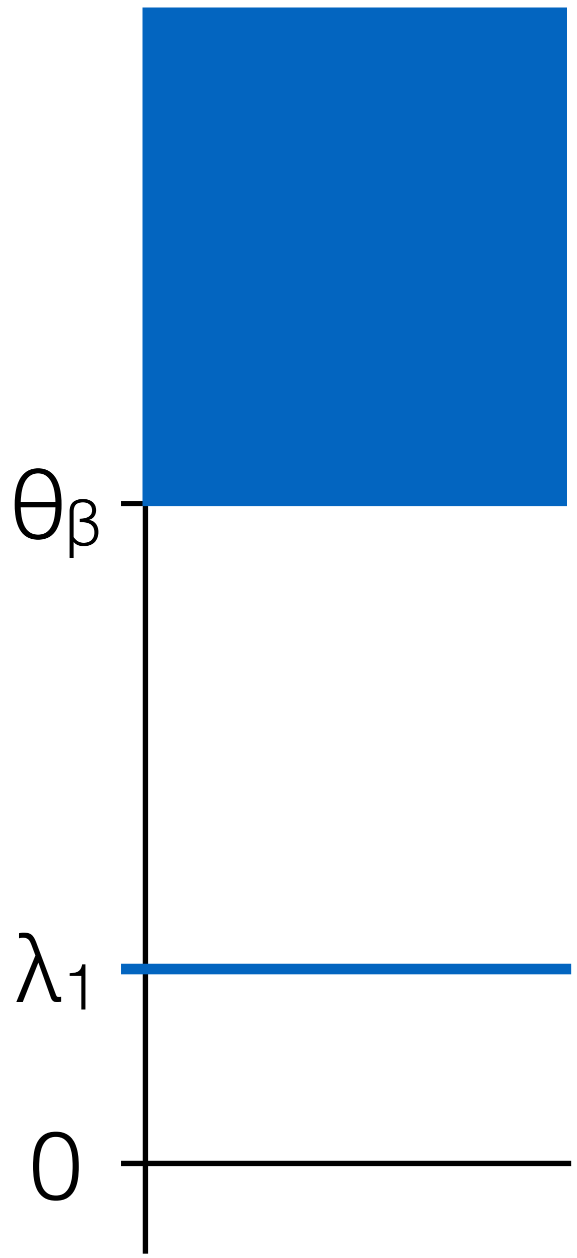


Magnetic Laplacian on a wedge

- Bottoms of essential spectra θ_β found by variational characterisation
- Bound state λ below θ_β?
  - Min-max principle:

$$\lambda_1 = \inf_{u \in H_A(\Omega)} \frac{\int_\Omega dS\, |\nabla_{\mathbf{A}} u|^2 - \beta \int_\Gamma dr\, |u|^2}{\int_\Omega dS\, |u|^2}$$

$$= \inf_{u \in H_A(\Omega)} R_{\mathbf{A}}(u)$$



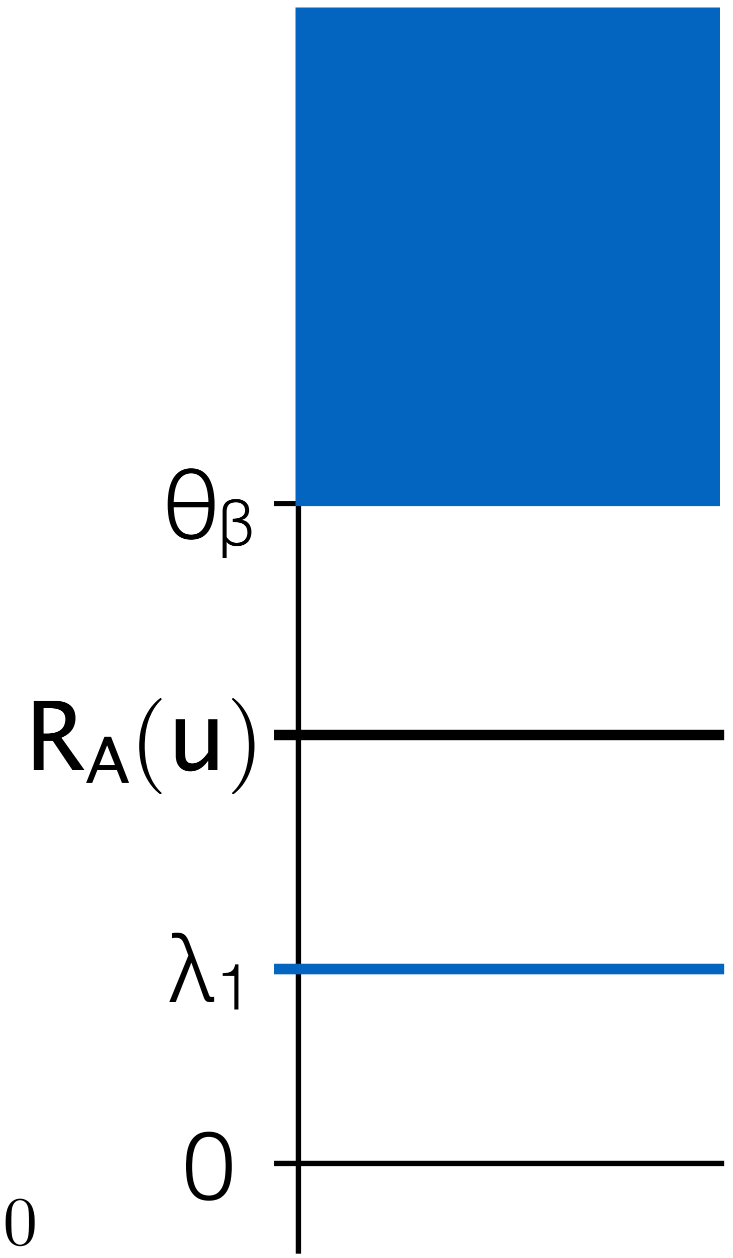
Magnetic Laplacian on a wedge

- Bottoms of essential spectra θ_β found by variational characterisation
- Bound state λ below θ_β?
  - Min-max principle:

$$\lambda_1 = \inf_{u \in H_A(\Omega)} \frac{\int_\Omega dS\, |\nabla_{\mathbf{A}} u|^2 - \beta \int_\Gamma dr\, |u|^2}{\int_\Omega dS\, |u|^2} = \inf_{u \in H_A(\Omega)} R_{\mathbf{A}}(u)$$

  - Find u such that $\mathbf{R_A(u)} < \theta_\beta$, i.e.

$$\mathcal{I}[u] = \int_\Omega dS \left(|\nabla_{\mathbf{A}} u|^2 - \theta_\beta |u|^2 \right) - \beta \int_\Gamma dr\, |u|^2 < 0$$



- **Ginzburg-Landau functional:**

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(|(-i\nabla + \kappa \mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 + \kappa^2 |\nabla \times \mathbf{A} - H|^2 \right)$$

$|\boldsymbol{\psi}|$ is the density of cooper pairs, **A** the magnetic potential, $\boldsymbol{\kappa}$ a physical constant, and H=$\nabla\times\mathbf{A_n}$ the applied (constant) magnetic field

See e.g. S. Fournais, B. Helffer, Spectral Methods in Surface Superconductivity, Birkhäuser, 2010

- **Ginzburg-Landau functional:**

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(|(-i\nabla + \kappa\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 + \kappa^2|\nabla \times \mathbf{A} - H|^2 \right)$$

$|\boldsymbol{\psi}|$ is the density of cooper pairs, **A** the magnetic potential, $\boldsymbol{\kappa}$ a physical constant, and $\mathbf{H}=\nabla\times\mathbf{A_n}$ the applied (constant) magnetic field

- For large $\boldsymbol{\kappa}$ (type 2 superconductor) and **H**, $\mathcal{G}$ is minimum at $|\psi|=\mathbf{0}$, $\mathbf{A}=\mathbf{A_n}$
(No superconductivity at large enough magnetic fields)

See e.g. S. Fournais, B. Helffer, Spectral Methods in Surface Superconductivity, Birkhäuser, 2010

- **Ginzburg-Landau functional:**

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(|(-i\nabla + \kappa\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 + \kappa^2|\nabla \times \mathbf{A} - H|^2 \right)$$

$|\boldsymbol{\psi}|$ is the density of cooper pairs, **A** the magnetic potential, $\boldsymbol{\kappa}$ a physical constant, and $\mathbf{H}=\nabla\times\mathbf{A_n}$ the applied (constant) magnetic field

- For large $\boldsymbol{\kappa}$ (type 2 superconductor) and **H**, $\mathcal{G}$ is minimum at $|\psi|=\mathbf{0}$, $\mathbf{A}=\mathbf{A_n}$
  (No superconductivity at large enough magnetic fields)

- Onset of superconductivity? Expanding near the normal state,

$$\mathcal{G}(\phi, \mathbf{A}_n + \mathbf{A}) \simeq constant + \int_\Omega \left(|(-i\nabla + \kappa\mathbf{A}_n)\phi|^2 - \kappa|\phi|^2 \right)$$

S. Fournais, B. Helffer, Spectral Methods in Surface Superconductivity, Birkhäuser, 2010

- **Ginzburg-Landau functional:**

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega \left(|(-i\nabla + \kappa \mathbf{A})\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 + \kappa^2 |\nabla \times \mathbf{A} - H|^2 \right)$$

$|\boldsymbol{\psi}|$ is the density of cooper pairs, **A** the magnetic potential, $\boldsymbol{\kappa}$ a physical constant, and $\mathbf{H}=\nabla\times\mathbf{A}_\mathbf{n}$ the applied (constant) magnetic field

- For large $\boldsymbol{\kappa}$ (type 2 superconductor) and **H**, $\mathcal{G}$ is minimum at $|\psi|=\mathbf{0}$, $\mathbf{A}=\mathbf{A}_\mathbf{n}$
  (No superconductivity at large enough magnetic fields)

- Onset of superconductivity? Expanding near the normal state,

$$\mathcal{G}(\phi, \mathbf{A}_n + \mathbf{A}) \simeq constant + \int_\Omega \left(|(-i\nabla + \kappa \mathbf{A}_n)\phi|^2 - \kappa |\phi|^2 \right)$$

- The normal state is not minimum if $\int_\Omega \left(|(-i\nabla + \kappa \mathbf{A}_n)\phi|^2 - \kappa |\phi|^2 \right) < 0$

→Together with a boundary condition we recover our bound state problem

S. Fournais, B. Helffer, Spectral Methods in Surface Superconductivity, Birkhäuser, 2010

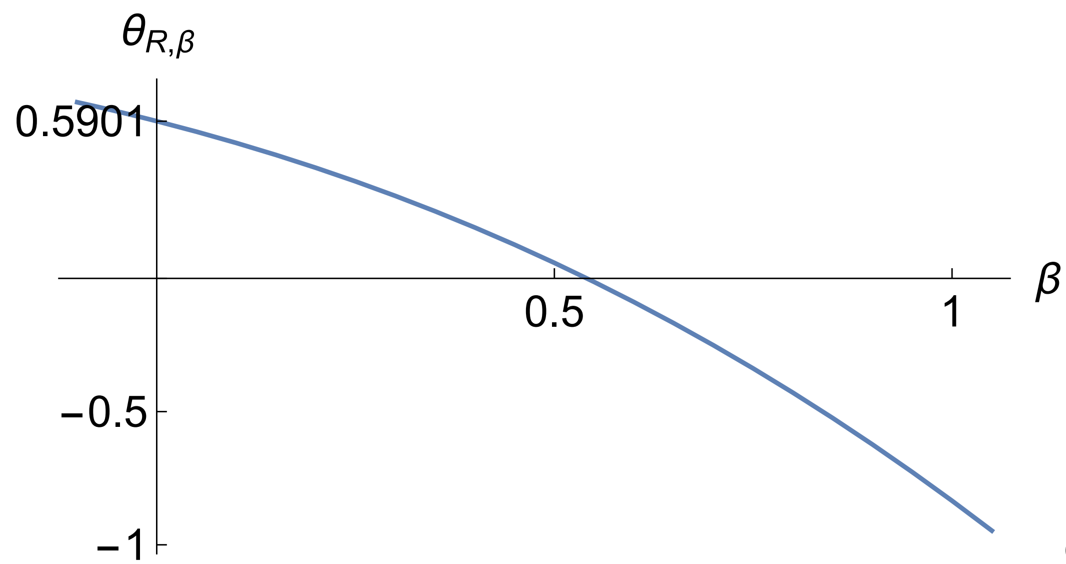
Neumann & Robin

- Find **u** that makes $\mathcal{I}[u] = \int_\Omega dS \left(|\nabla_{\mathbf{A}} u|^2 - \theta_{R,\beta} |u|^2 \right) - \beta \int_\Gamma dr\, |u|^2 < 0$

Neumann & Robin

- Find **u** that makes $\mathcal{I}[u] = \int_\Omega dS \left(|\nabla_{\mathbf{A}} u|^2 - \theta_{R,\beta}\, |u|^2 \right) - \beta \int_\Gamma dr\, |u|^2 < 0$
- $\theta_{\mathsf{R},\beta}$ is the bottom of the essential spectrum:

$$\theta_{\mathrm{R},\beta} := \inf_{r_0,\eta} \frac{\int_{r_0}^{\infty} \left(|g'(r)|^2 + r^2 |g(r)|^2\right) \mathsf{d}r - \beta |g(r_0)|^2}{\int_{r_0}^{\infty} |g(r)|^2 \mathsf{d}r}$$, with $-g''(r) + r^2 g(r) = \eta g(r)$



$\Theta_{R,\beta} \geq -\beta^2$

- We try functions of the type $u_\star(r,\theta)=e^{-ar^2/2}\,e^{ib(r,\theta)}$ with $b(r,\theta)=r\,b_1(\theta)+r^2\,b_2(\theta)+r^3\,b_3(\theta)+\ldots$

- We try functions of the type $u_\star(r,\theta)=e^{-ar^2/2}e^{ib(r,\theta)}$ with $b(r,\theta)=r\,b_1(\theta)+r^2\,b_2(\theta)+r^3\,b_3(\theta)+\ldots$

- We start with $b(r,\theta)=r\,b_1(\theta)$ and systematically improve:

$$\mathcal{I}[u_\star]=\frac{1}{2a}\int_0^\phi\left(b_1^2+(\partial_\theta b_1)^2-\frac{\sqrt{\pi}}{2\sqrt{a}}(\partial_\theta b_1)\right)\mathrm{d}\theta+\frac{\phi}{2}-\frac{\Theta_{\mathrm{R},\beta}\phi}{2a}+\frac{\phi}{8a^2}-\beta\sqrt{\frac{\pi}{a}}$$

- We try functions of the type $u_\star(r,\theta)=e^{-ar^2/2}e^{ib(r,\theta)}$ with $b(r,\theta)=r\,b_1(\theta)+r^2\,b_2(\theta)+r^3\,b_3(\theta)+\dots$

- We start with $b(r,\theta)=r\,b_1(\theta)$ and systematically improve:

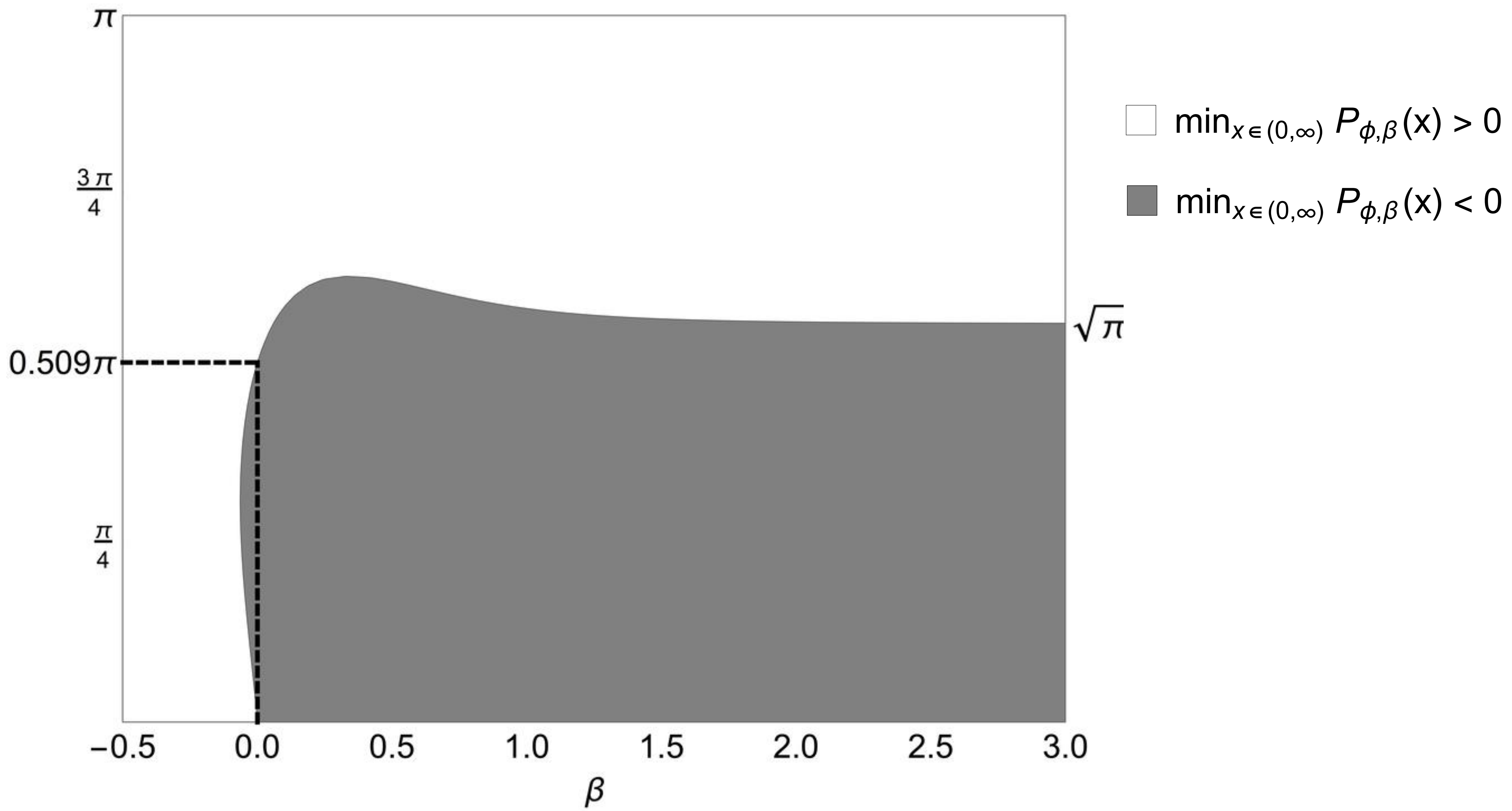
$$\mathcal{I}[u_\star]=\frac{1}{2a}\int_0^\phi\left(b_1^2+(\partial_\theta b_1)^2-\frac{\sqrt{\pi}}{2\sqrt{a}}(\partial_\theta b_1)\right)\mathrm{d}\theta+\frac{\phi}{2}-\frac{\Theta_{\mathrm{R},\beta}\phi}{2a}+\frac{\phi}{8a^2}-\beta\sqrt{\frac{\pi}{a}}$$

- Optimal: $b_1(\theta)=\frac{\sqrt{\pi}x}{4(1+e^\phi)}\left(e^\theta-e^{-(\phi+\theta)}\right)$, with $x=\frac{1}{\sqrt{a}}$

$$\mathcal{I}[u_\star]=x^4\left(\frac{\phi}{8}-\frac{\pi\tanh(\phi/2)}{16}\right)-\frac{\Theta_{\mathrm{R},\beta}\phi x^2}{2}+\frac{\phi}{2}-\beta\sqrt{\pi}x=\frac{P_{\phi,\beta}(x)}{16}$$

$$\mathcal{I}[u_\star] = \int_\Omega dS \left(|\nabla_{\mathbf{A}} u_\star|^2 - \theta_{R,\beta} |u_\star|^2 \right) - \beta \int_\Gamma dr\, |u_\star|^2$$
$$= x^4 \left(\frac{\phi}{8} - \frac{\pi \tanh(\phi/2)}{16} \right) - \frac{\Theta_{\mathrm{R},\beta}\phi x^2}{2} + \frac{\phi}{2} - \beta\sqrt{\pi}x = \frac{P_{\phi,\beta}(x)}{16}$$

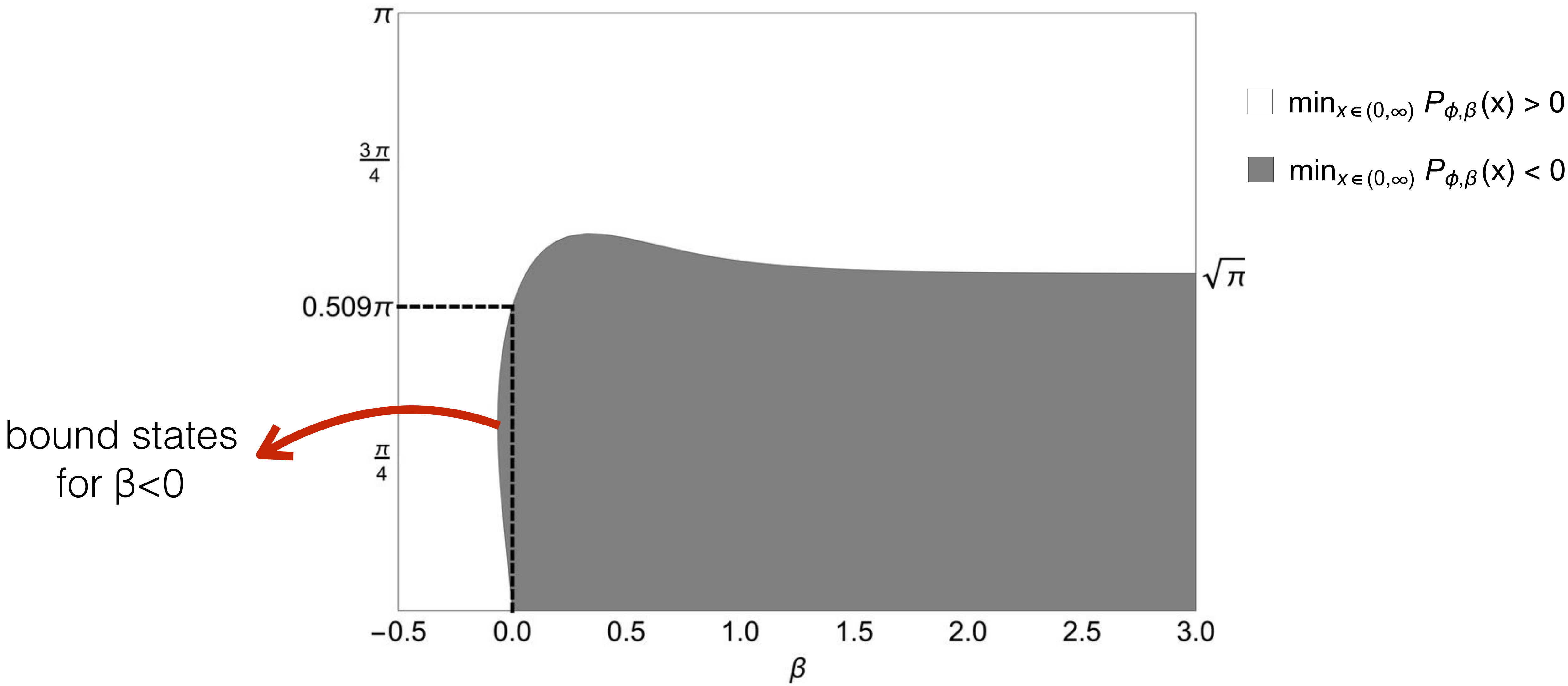
with $u_\star = e^{-r^2/(2x^2)} e^{i\, r\, b_1(\theta)}$



$$\mathcal{I}[u_\star] = \int_\Omega dS \left(|\nabla_{\mathbf{A}} u_\star|^2 - \theta_{R,\beta} |u_\star|^2 \right) - \beta \int_\Gamma dr \, |u_\star|^2$$

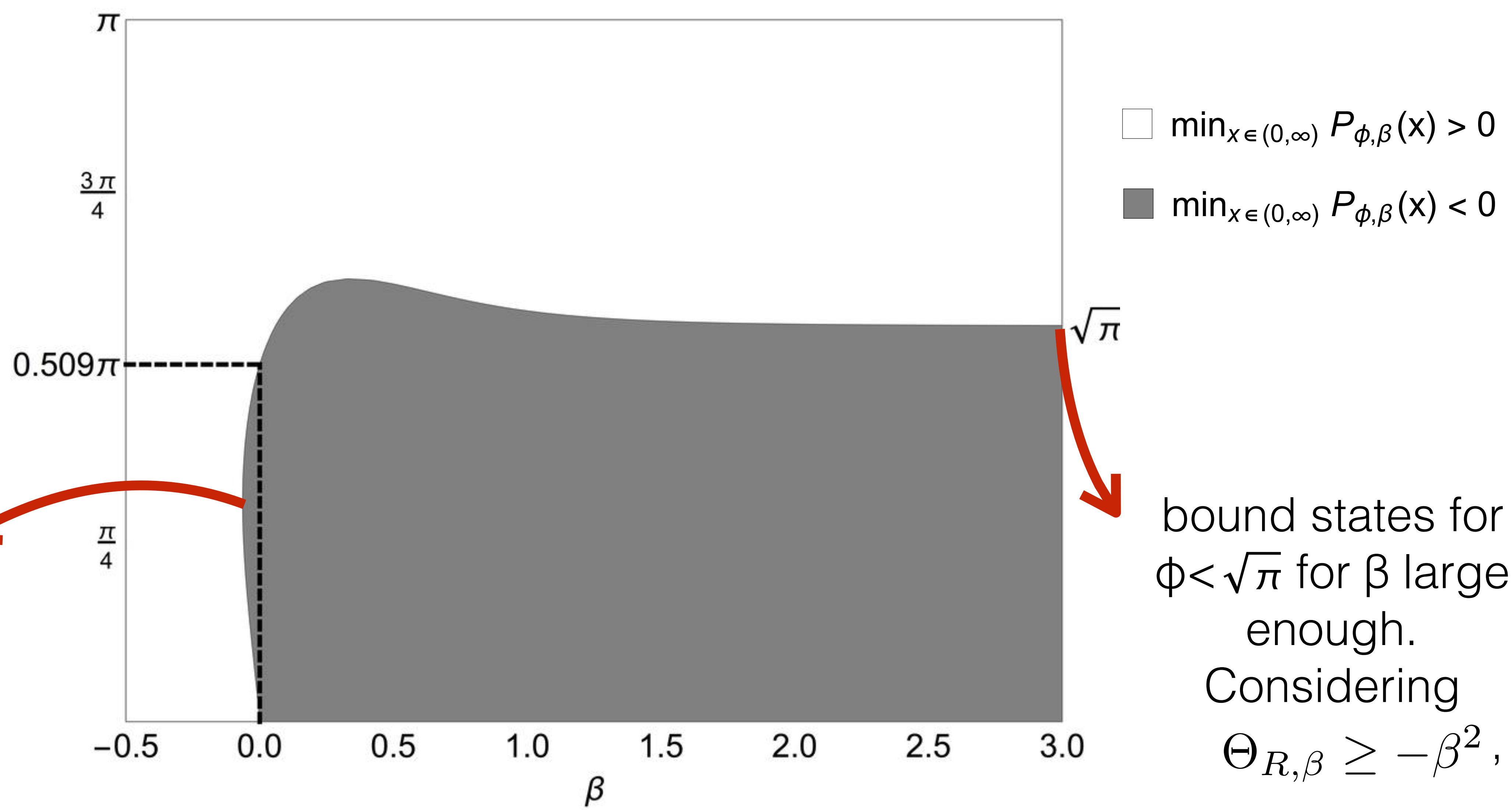
$$= x^4 \left(\frac{\phi}{8} - \frac{\pi \tanh(\phi/2)}{16} \right) - \frac{\Theta_{\mathrm{R},\beta} \phi x^2}{2} + \frac{\phi}{2} - \beta \sqrt{\pi} x = \frac{P_{\phi,\beta}(x)}{16}$$

with $u_\star = e^{-r^2/(2x^2)} e^{i\, r\, b_1(\theta)}$



$$\mathcal{I}[u_\star] = \int_\Omega dS \left(|\nabla_{\mathbf{A}} u_\star|^2 - \theta_{R,\beta} |u_\star|^2 \right) - \beta \int_\Gamma dr\, |u_\star|^2$$
$$= x^4 \left(\frac{\phi}{8} - \frac{\pi \tanh(\phi/2)}{16} \right) - \frac{\Theta_{\mathrm{R},\beta} \phi x^2}{2} + \frac{\phi}{2} - \beta\sqrt{\pi} x = \frac{P_{\phi,\beta}(x)}{16}$$

with $u_\star = e^{-r^2/(2x^2)} e^{i\, r\, b_1(\theta)}$



bound states for $\beta<0$

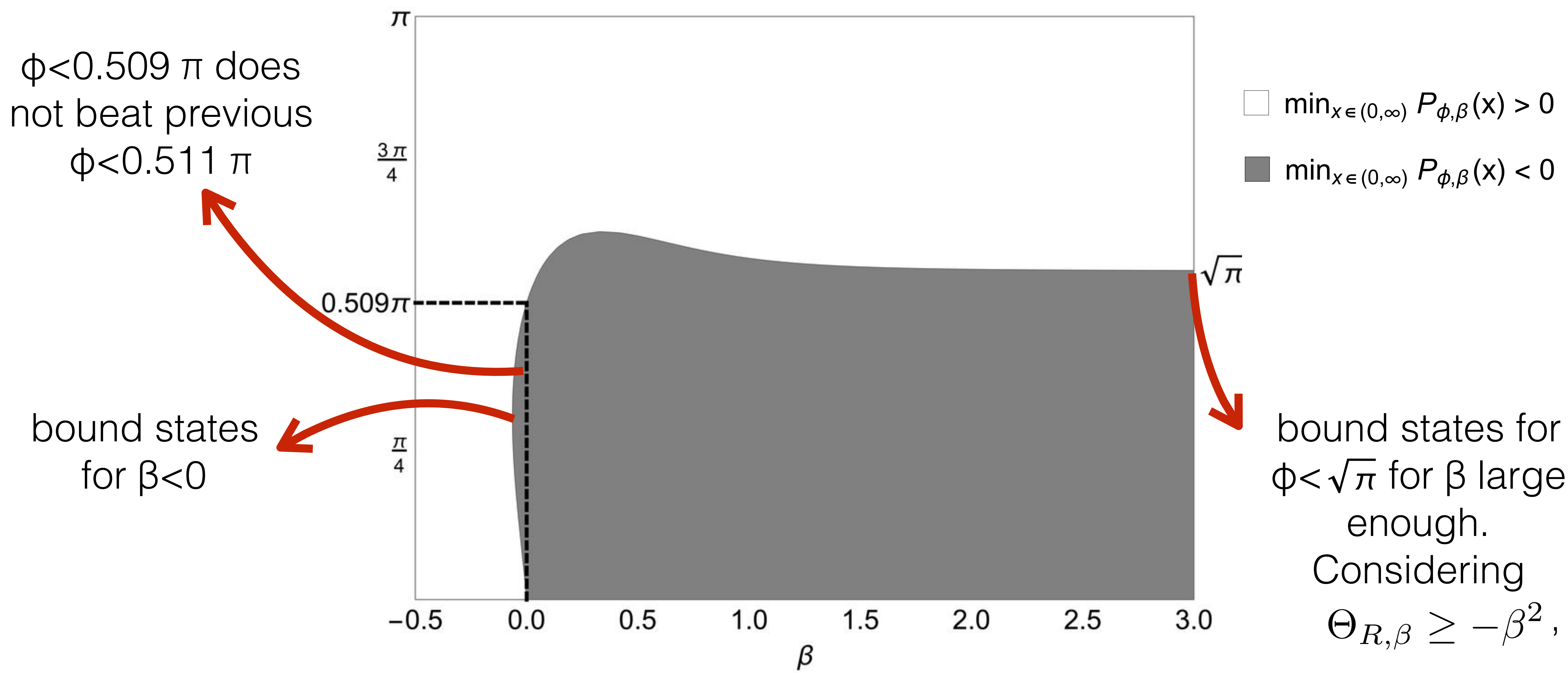
bound states for $\phi<\sqrt{\pi}$ for β large enough. Considering $\Theta_{R,\beta} \geq -\beta^2$,

$$P_{\phi,\beta}(1/\beta) \leq \beta^{-4} \left(2\phi - \pi \tanh\left(\frac{\phi}{2}\right) \right) + 16 \left(\phi - \sqrt{\pi}\right) < 0$$

$$\mathcal{I}[u_\star] = \int_\Omega dS\left(|\nabla_{\mathbf{A}} u_\star|^2 - \theta_{R,\beta}|u_\star|^2\right) - \beta\int_\Gamma dr\,|u_\star|^2$$

$$= x^4\left(\frac{\phi}{8} - \frac{\pi\tanh(\phi/2)}{16}\right) - \frac{\Theta_{\mathrm{R},\beta}\phi x^2}{2} + \frac{\phi}{2} - \beta\sqrt{\pi}x = \frac{P_{\phi,\beta}(x)}{16}$$

with $u_\star = e^{-r^2/(2x^2)} e^{i\,r\,b_1(\theta)}$



Considering $\Theta_{R,\beta} \geq -\beta^2$,

$$P_{\phi,\beta}(1/\beta) \leq \beta^{-4}\left(2\phi - \pi\tanh\left(\frac{\phi}{2}\right)\right) + 16\left(\phi - \sqrt{\pi}\right) < 0$$

Improvements in Neumann

- Now we try $u_\star(r,\theta)=e^{-a/2\cdot r^2}e^{i(r\cdot b_1(\theta)+r^2\cdot b_2(\theta))}$ and obtain:

$$\mathcal{I}[u_\star]=\int_0^\phi\left[\frac{(b_1'(\theta))^2}{2a}+\frac{(b_2'(\theta))^2}{2a^2}+\frac{\sqrt{\pi}b_1'(\theta)b_2'(\theta)}{2a^{3/2}}+\frac{(b_1(\theta))^2}{2a}+\frac{2(b_2(\theta))^2}{a^2}\right.$$
$$\left.+\frac{\sqrt{\pi}b_1(\theta)b_2(\theta)}{a^{3/2}}-\frac{\sqrt{\pi}b1'(\theta)}{4a^{3/2}}-\frac{b2'(\theta)}{2a^2}\right]\mathrm{d}\theta+\frac{\phi(4a^2+1-4a\Theta_0)}{8a^2}$$

Improvements in Neumann

- Now we try $u_\star(r,\theta)=e^{-a/2\cdot r^2}e^{i(r\cdot b_1(\theta)+r^2\cdot b_2(\theta))}$ and obtain:

$$\mathcal{I}[u_\star] = \int_0^\phi \left[\frac{(b_1'(\theta))^2}{2a} + \frac{(b_2'(\theta))^2}{2a^2} + \frac{\sqrt{\pi} b_1'(\theta) b_2'(\theta)}{2a^{3/2}} + \frac{(b_1(\theta))^2}{2a} + \frac{2(b_2(\theta))^2}{a^2} + \frac{\sqrt{\pi} b_1(\theta) b_2(\theta)}{a^{3/2}} - \frac{\sqrt{\pi} b1'(\theta)}{4a^{3/2}} - \frac{b2'(\theta)}{2a^2} \right] \mathrm{d}\theta + \frac{\phi(4a^2+1-4a\Theta_0)}{8a^2}$$

- Optimal b_1 and b_2:
  - Functional derivative: $-\begin{pmatrix} 2a & \sqrt{a\pi} \\ \sqrt{a\pi} & 2 \end{pmatrix} \begin{pmatrix} b_1''(\theta) \\ b_2''(\theta) \end{pmatrix} + \begin{pmatrix} 2a & 2\sqrt{a\pi} \\ 2\sqrt{a\pi} & 8 \end{pmatrix} \begin{pmatrix} b_1(\theta) \\ b_2(\theta) \end{pmatrix} = 0$
  - Usual derivate with respect to the free parameters

Improvements in Neumann

- Now we try $u_\star(r,\theta)=e^{-a/2\cdot r^2}e^{i(r\cdot b_1(\theta)+r^2\cdot b_2(\theta))}$ and obtain:

$$\mathcal{I}[u_\star]=\int_0^\phi\left[\frac{(b_1'(\theta))^2}{2a}+\frac{(b_2'(\theta))^2}{2a^2}+\frac{\sqrt{\pi}b_1'(\theta)b_2'(\theta)}{2a^{3/2}}+\frac{(b_1(\theta))^2}{2a}+\frac{2(b_2(\theta))^2}{a^2}\right.$$
$$\left.+\frac{\sqrt{\pi}b_1(\theta)b_2(\theta)}{a^{3/2}}-\frac{\sqrt{\pi}b1'(\theta)}{4a^{3/2}}-\frac{b2'(\theta)}{2a^2}\right]\mathrm{d}\theta+\frac{\phi(4a^2+1-4a\Theta_0)}{8a^2}$$

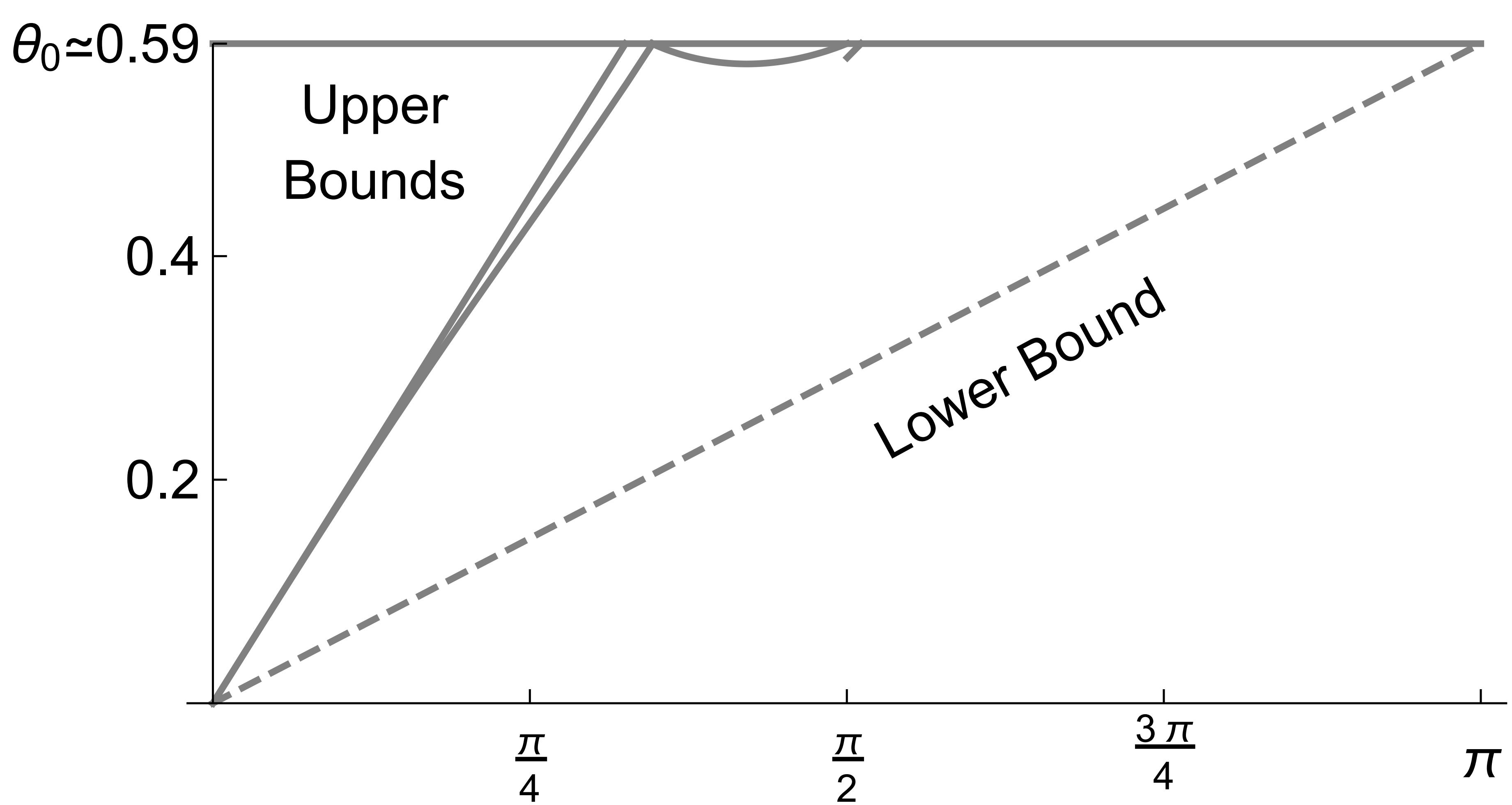
- Optimal b_1 and b_2:
  - Functional derivative: $-\begin{pmatrix}2a & \sqrt{a\pi}\\ \sqrt{a\pi} & 2\end{pmatrix}\begin{pmatrix}b_1''(\theta)\\ b_2''(\theta)\end{pmatrix}+\begin{pmatrix}2a & 2\sqrt{a\pi}\\ 2\sqrt{a\pi} & 8\end{pmatrix}\begin{pmatrix}b_1(\theta)\\ b_2(\theta)\end{pmatrix}=0$
  - Usual derivate with respect to the free parameters

- $\mathcal{I}[u_\star]=\dfrac{\phi}{2}-\phi^2 s\Theta_0^2\left[2\phi s-\mu_1^2\mu_2^2\left\{\nu_1\tanh\left(\tfrac{1}{2}\mu_1\phi\right)+\nu_2\tanh\left(\tfrac{1}{2}\mu_2\phi\right)\right\}\right]^{-1}$

$$\nu_{1,2}=\frac{\sqrt{4-\pi}}{2}\frac{3-\pi\pm s}{1\pm s},\quad s=\sqrt{9-2\pi},\quad \mu_{1,2}=\frac{s\pm1}{\sqrt{4-\pi}}$$

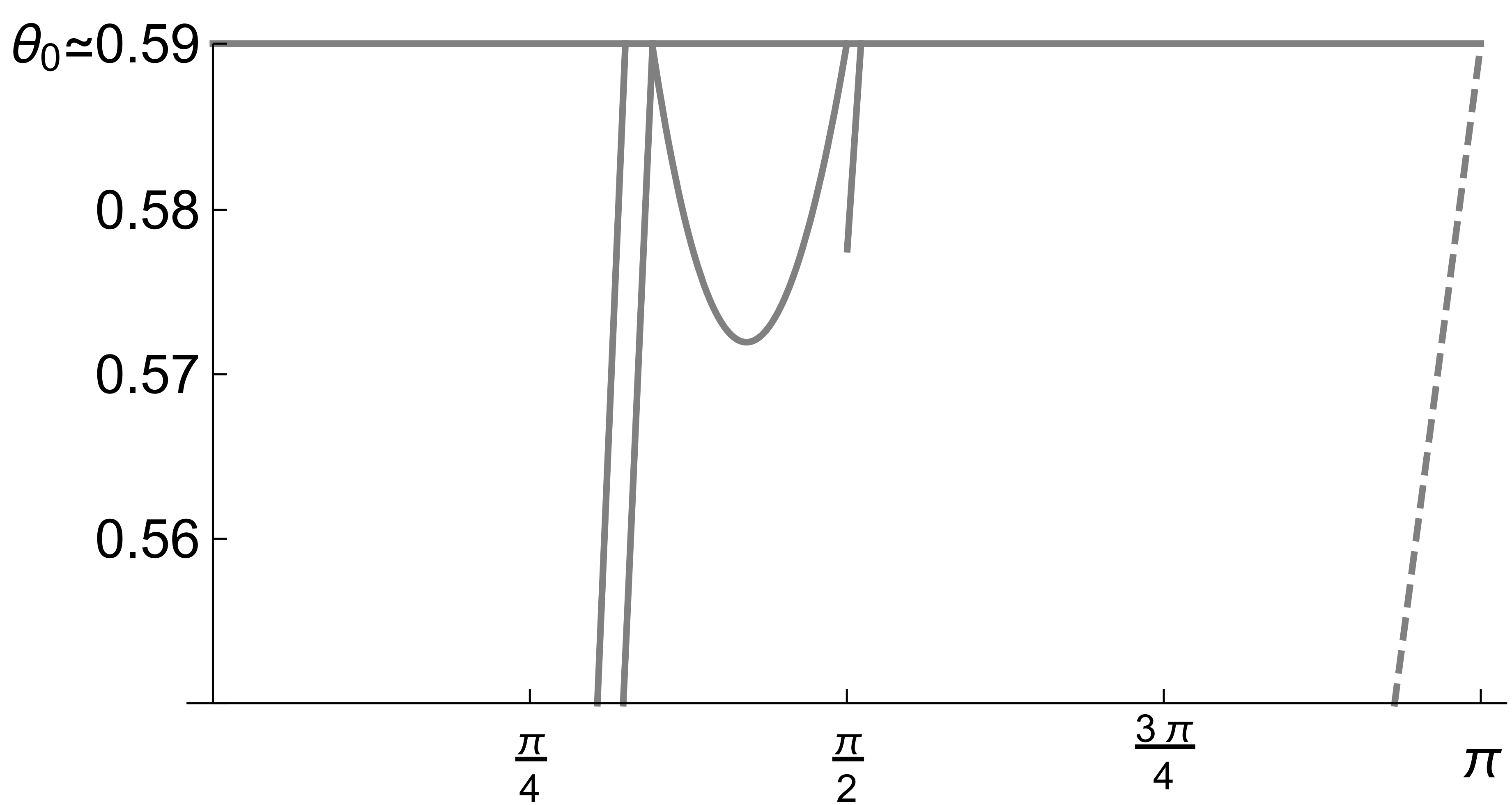
→ **$\phi<0.583\pi$!**

Comparison of Neumann results



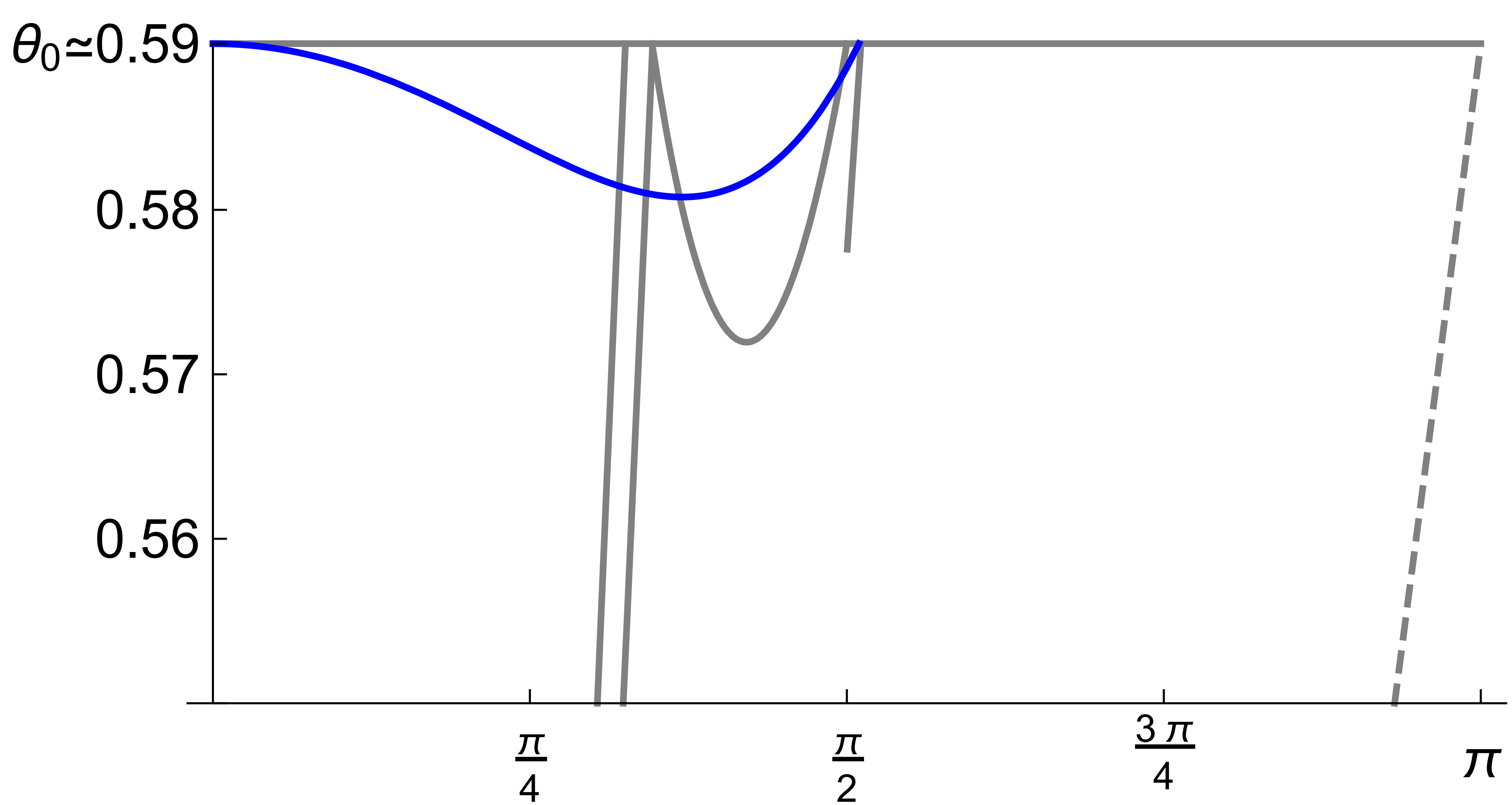
Review of previous results from
V. Bonnaillie, Thèse de doctorat,
Université Paris XI (2003)

Comparison of Neumann results



Review of previous results from
V. Bonnaillie, Thèse de doctorat,
Université Paris XI (2003)

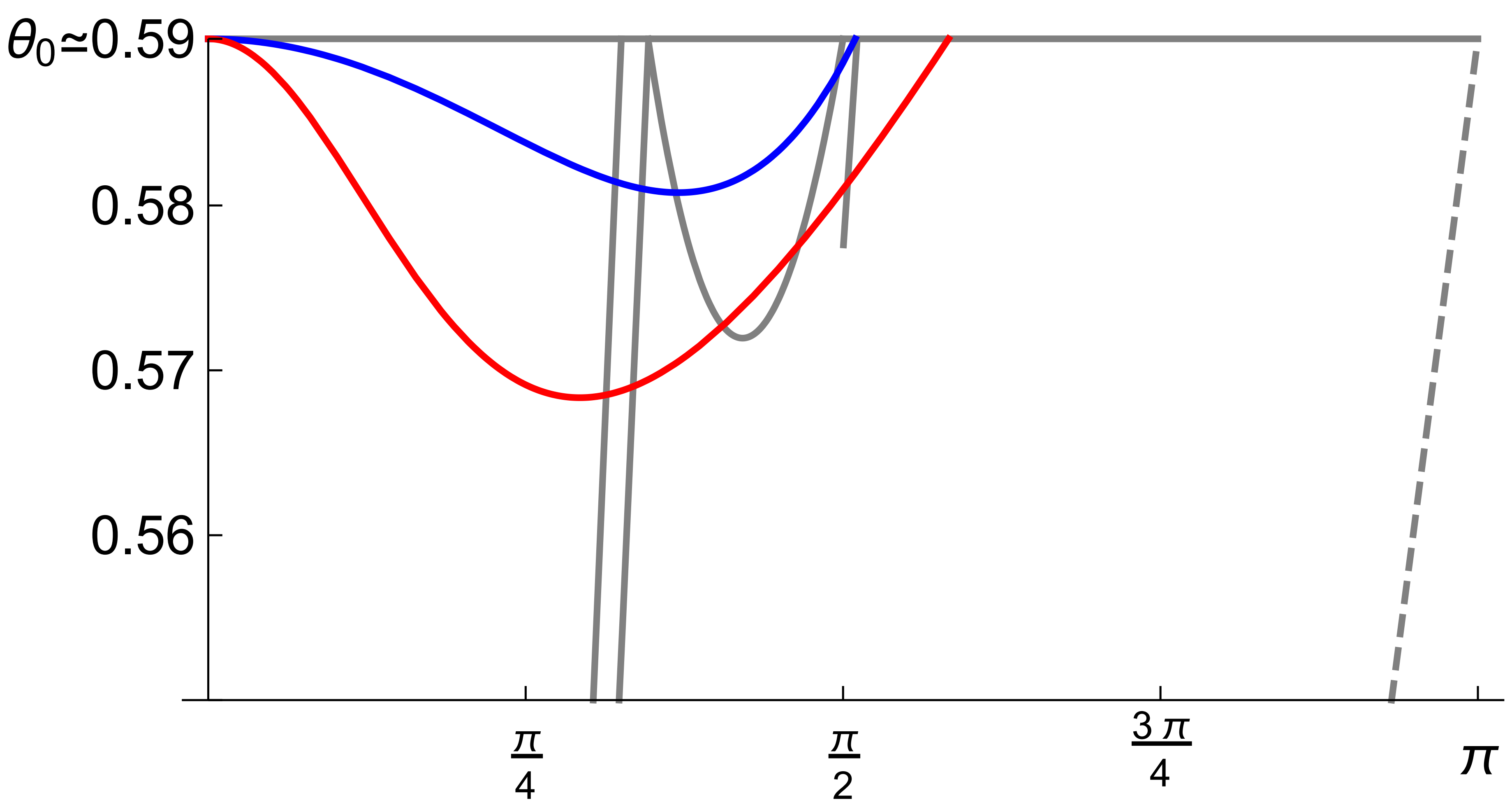
Comparison of Neumann results



Review of previous results from V. Bonnaillie, Thèse de doctorat, Université Paris XI (2003)

$b(r,\theta) = r \cdot b_1(\theta)$

Comparison of Neumann results

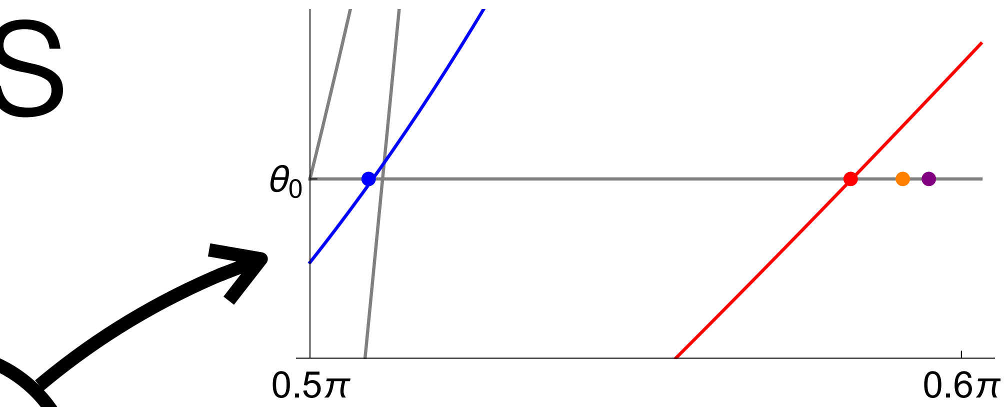


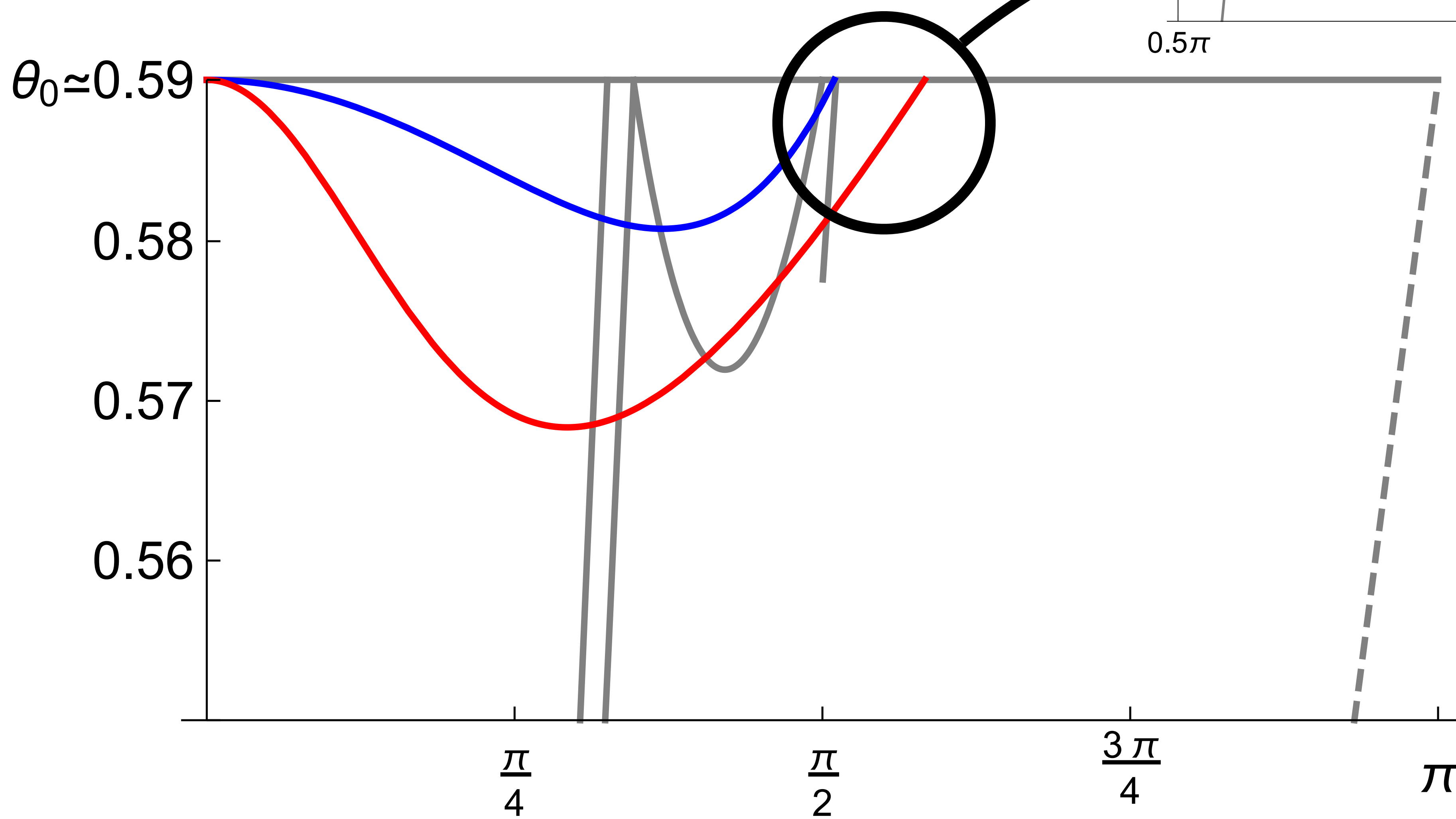
Review of previous results from V. Bonnaillie, Thèse de doctorat, Université Paris XI (2003)

$b(r,\theta) = r \cdot b_1(\theta)$

$b(r,\theta) = r \cdot b_1(\theta) + r^2 \cdot b_2(\theta)$

Comparison of Neumann results





Review of previous results from V. Bonnaillie, Thèse de doctorat, Université Paris XI (2003)

$b(r,\theta) = r \cdot b_1(\theta)$

$b(r,\theta) = r \cdot b_1(\theta) + r^2 \cdot b_2(\theta)$

$b(r,\theta) = r \cdot b_1(\theta) + r^2 \cdot b_2(\theta) + r^3 \cdot b_3(\theta)$

$b(r,\theta) = r \cdot b_1(\theta) + r^2 \cdot b_2(\theta) + r^3 \cdot b_3(\theta) + r^4 \cdot b_4(\theta)$

Delta interaction

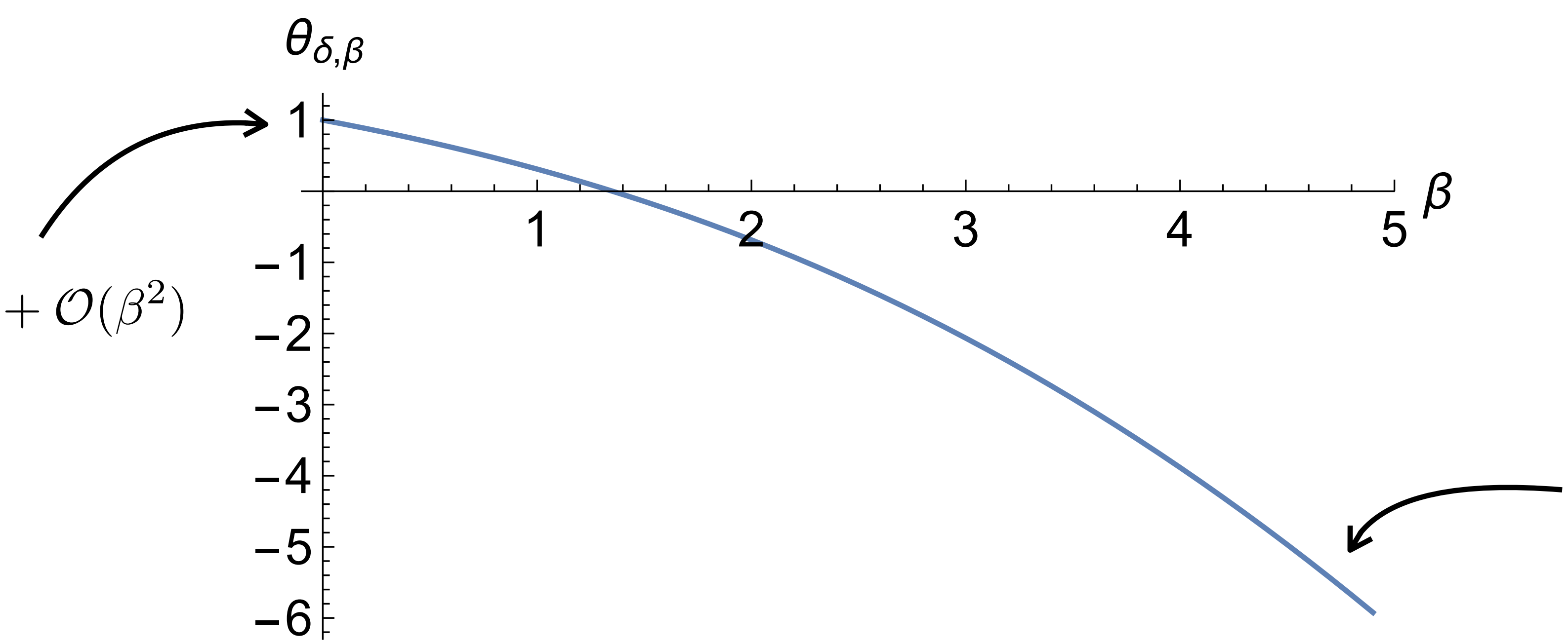
- Find **u** that makes $\mathcal{I}[u] = \int_{\mathbb{R}^2} dS \left(|\nabla_{\mathbf{A}} u|^2 - \theta_{\delta,\beta} |u|^2 \right) - \beta \int_{\Gamma} dr\, |u|^2 < 0$

Delta interaction

- Find **u** that makes $\mathcal{I}[u] = \int_{\mathbb{R}^2} dS \left(|\nabla_{\mathbf{A}} u|^2 - \theta_{\delta,\beta} |u|^2 \right) - \beta \int_{\Gamma} dr\, |u|^2 < 0$
- The bottom of the essential spectrum $\theta_{\delta,\beta}$ is

$$\theta_{\delta,\beta} := \inf_{r_0,\eta} \frac{\int_{-\infty}^{\infty} \left(|g'(r)|^2 + r^2 |g(r)|^2 \right) \mathrm{d}r - \beta |g(r_0)|^2}{\int_{-\infty}^{\infty} |g(r)|^2 \mathrm{d}r}$$, with $-g''(r) + r^2 g(r) = \eta g(r)$

$\Theta_{\delta,\beta} = 1 - \frac{\beta}{\sqrt{\pi}} + \mathcal{O}(\beta^2)$



$\Theta_{\delta,\beta} \geq -\frac{\beta^2}{4}$

- Useful to rotate **A** by π/4 and shift it *c*. Equivalently, we rotate and shift the wedge

$$\mathcal{I}[u_\star] := \int_0^{2\pi}\int_0^{\infty} \left[|\nabla u_\star|^2 + (|\mathbf{A}|^2 - \Theta_{\delta,\beta})|u_\star|^2\right] r\,\mathsf{d}r\,\mathsf{d}\theta$$
$$- \beta \int_0^{\infty} |u_\star(r\cos\phi_+ - c, r\sin\phi_+ - c)|^2 \mathsf{d}r - \beta \int_0^{\infty} |u_\star(r\cos\phi_- - c, r\sin\phi_- - c)|^2 \mathsf{d}r$$

with $\phi_\pm := \pi/4 \pm \phi/2$

- Useful to rotate **A** by π/4 and shift it *c*. Equivalently, we rotate and shift the wedge

$$\mathcal{I}[u_\star] := \int_0^{2\pi}\int_0^\infty \left[|\nabla u_\star|^2 + (|\mathbf{A}|^2 - \Theta_{\delta,\beta})|u_\star|^2\right] r\,\mathsf{d}r\,\mathsf{d}\theta$$
$$- \beta\int_0^\infty |u_\star(r\cos\phi_+ - c, r\sin\phi_+ - c)|^2\mathsf{d}r - \beta\int_0^\infty |u_\star(r\cos\phi_- - c, r\sin\phi_- - c)|^2\mathsf{d}r$$

with $\phi_\pm := \pi/4 \pm \phi/2$

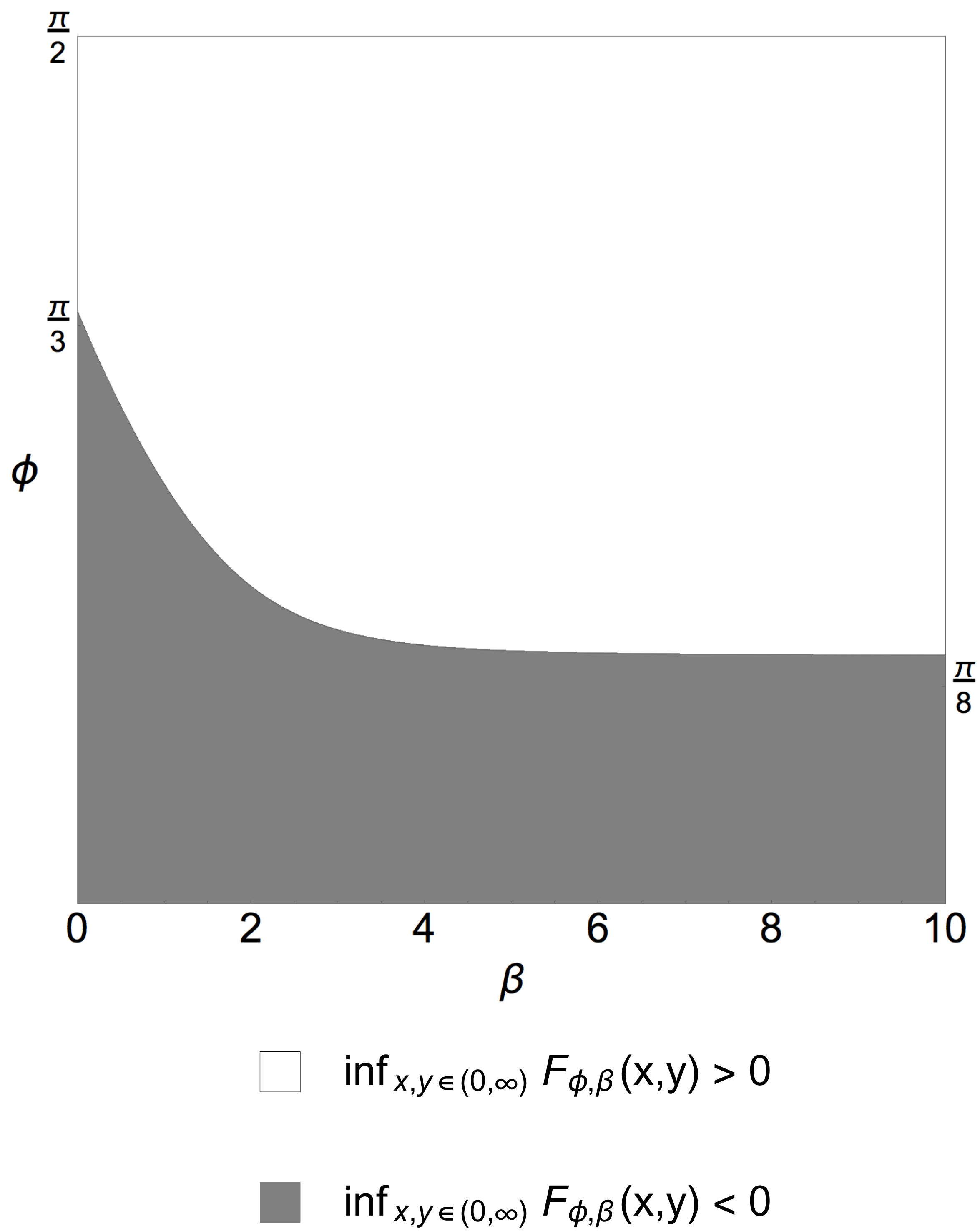
- Using $u_\star(r,\theta) = e^{-a/2\cdot r^2}$,

$$\mathcal{I}[u_\star] = \pi\left(1 + \frac{x^4}{4} - x^2\Theta_{\delta,\beta}\right) - \beta x\sqrt{\pi}e^{-y^2\tan^2(\phi/2)}\left(1 + \mathrm{erf}\,(y)\right) = \pi F_{\phi,\beta}(x,y)$$

with $x = 1/\sqrt{a}$ and $y = \sqrt{2ac^2}\cos(\phi/2)$

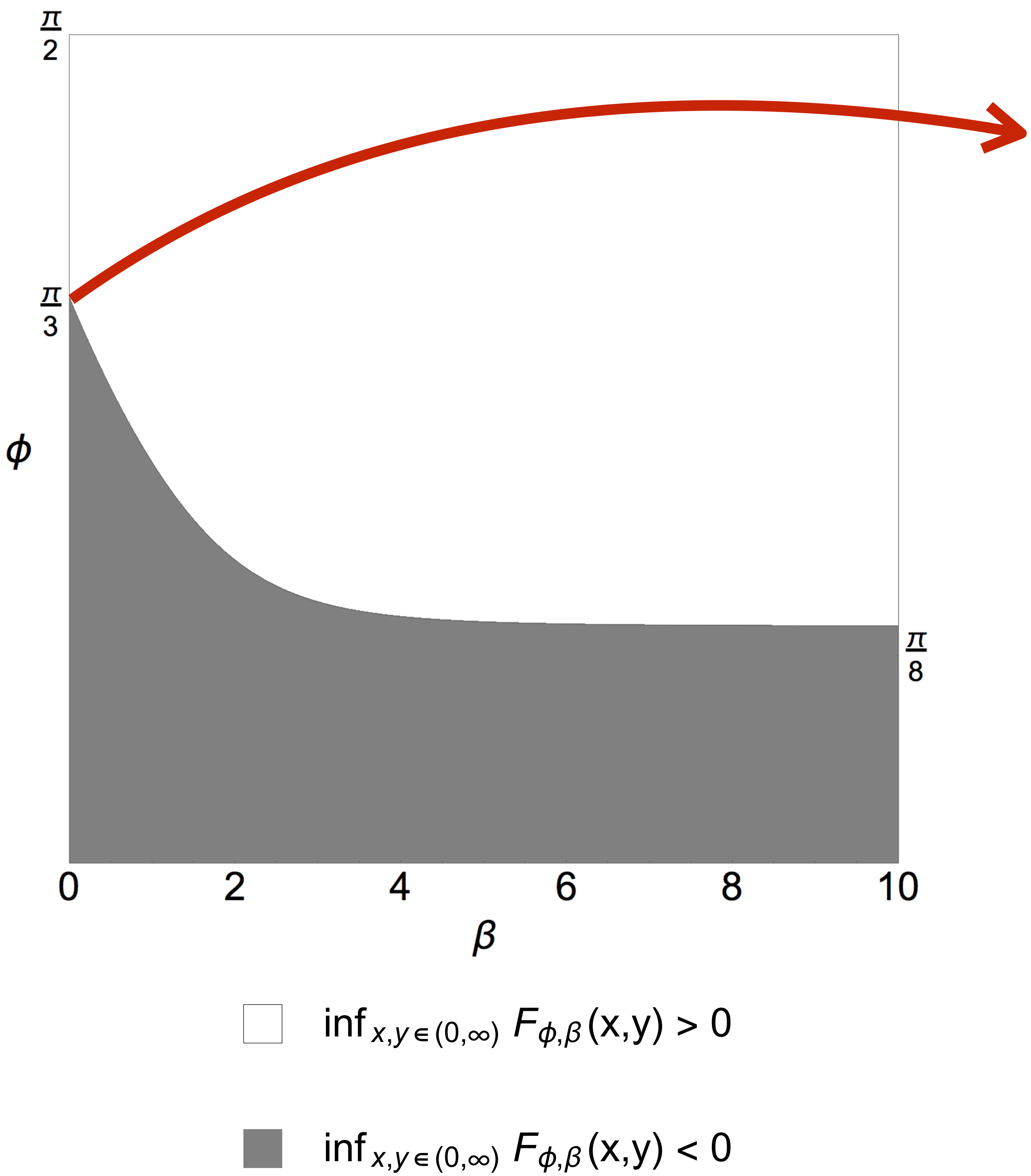
$$\mathcal{I}[u_\star] = \pi\left(1 + \frac{x^4}{4} - x^2\Theta_{\delta,\beta}\right) - \beta x\sqrt{\pi}e^{-y^2\tan^2(\phi/2)}\left(1 + \mathrm{erf}\left(y\right)\right) = \pi F_{\phi,\beta}(x,y)$$

with $u_\star = e^{-r^2/(2x^2)}$



$$\mathcal{I}[u_\star] = \pi\left(1 + \frac{x^4}{4} - x^2\Theta_{\delta,\beta}\right) - \beta x\sqrt{\pi}e^{-y^2\tan^2(\phi/2)}\left(1 + \mathrm{erf}\left(y\right)\right) = \pi F_{\phi,\beta}(x,y)$$

with $u_\star = e^{-r^2/(2x^2)}$



As β → 0+

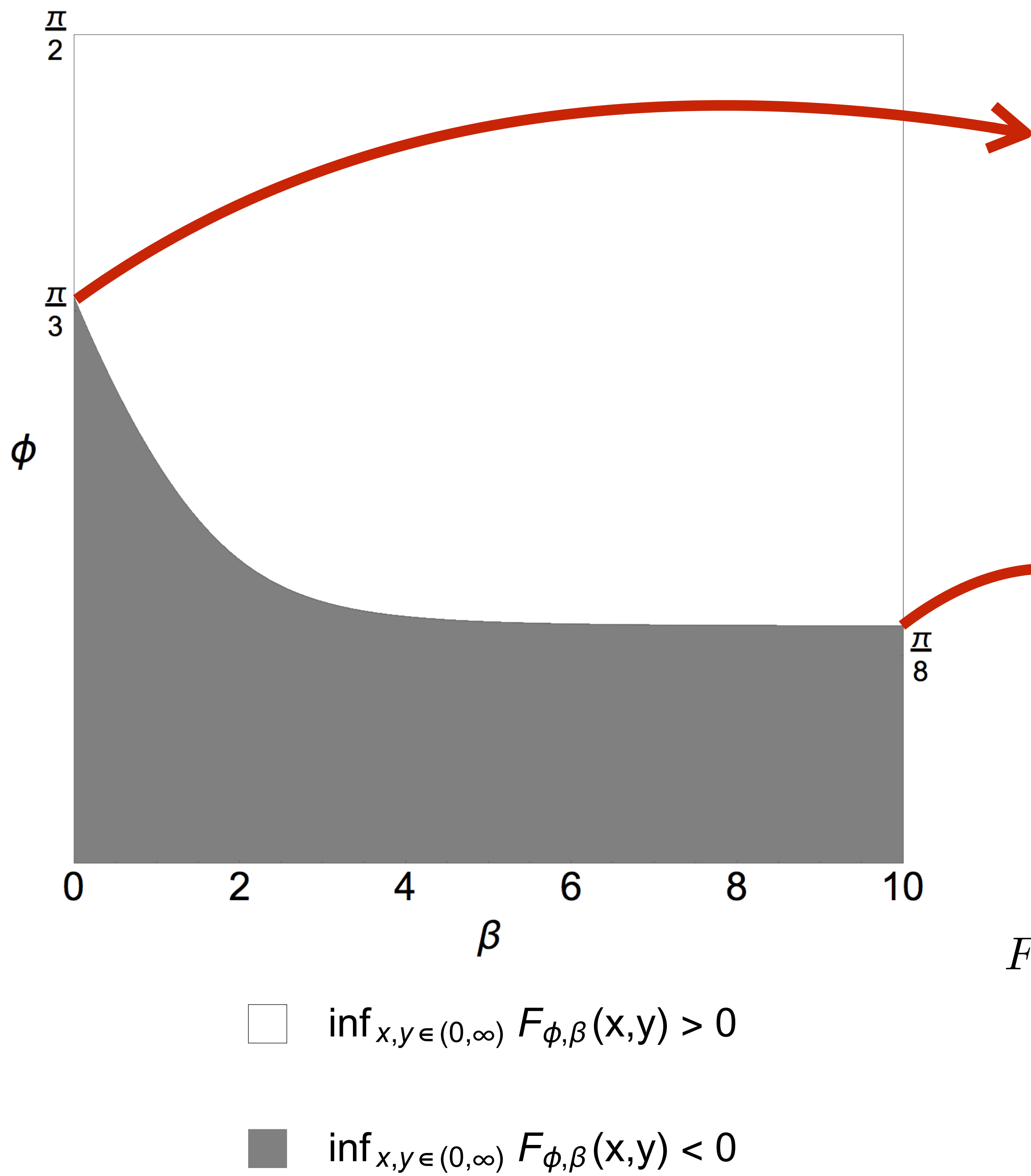
$$\Theta_{\delta,\beta} = 1 - \frac{\beta}{\sqrt{\pi}} + \mathcal{O}(\beta^2)$$

For $\phi \in (0, \pi/3]$

$$F_{\phi,\beta}(\sqrt{2}, \tfrac{17\sqrt{3}}{40}) \leq F_{\frac{\pi}{3},\beta}(\sqrt{2}, \tfrac{17\sqrt{3}}{40}) \simeq -0.003\beta + \mathcal{O}(\beta^2)$$

$$\mathcal{I}[u_\star] = \pi\left(1 + \frac{x^4}{4} - x^2\Theta_{\delta,\beta}\right) - \beta x\sqrt{\pi}e^{-y^2\tan^2(\phi/2)}\left(1 + \mathrm{erf}\,(y)\right) = \pi F_{\phi,\beta}(x,y)$$

with $u_\star = e^{-r^2/(2x^2)}$



As $\beta \to 0+$

$$\Theta_{\delta,\beta} = 1 - \frac{\beta}{\sqrt{\pi}} + \mathcal{O}(\beta^2)$$

For $\phi \in (0, \pi/3]$

$$F_{\phi,\beta}(\sqrt{2}, \tfrac{17\sqrt{3}}{40}) \leq F_{\frac{\pi}{3},\beta}(\sqrt{2}, \tfrac{17\sqrt{3}}{40}) \simeq -0.003\beta + \mathcal{O}(\beta^2)$$

For large β

$$\Theta_{\delta,\beta} \geq -\frac{\beta^2}{4}$$

For $\phi \in (0, \pi/8]$

$$F_{\phi,\beta}\left(\frac{g_\phi(\frac{13}{10})}{\beta}, \frac{13}{10}\right) \leq 1 - \frac{g_{\frac{\pi}{8}}(\frac{13}{10})^2}{4} + \mathcal{O}(\beta^{-4}) \simeq -0.042 + \mathcal{O}(\beta^{-4})$$

$$g_\phi(y) := 2\pi^{-1/2}e^{-y^2\tan^2(\phi/2)}\left(1 + \mathrm{erf}\,(y)\right)$$

Summary & conclusions

- We have investigated the existence of bound states for the **Laplacian with a constant magnetic field on a wedge** with different boundary conditions and angles
- The **Neumann** case has been proved up to $\boldsymbol{\phi \lesssim 0.583\pi\ (\sim 105^o)}$, 14% more than the previous limit.
- For the **Robin** boundary condition we have proved the existence of bound states for $\boldsymbol{\phi \lesssim 0.564\pi\ (\sim 102^o)}$ **for positive β**. For small **negative β** there are also bound states for $\boldsymbol{\phi \lesssim 0.51\pi\ (92^o)}$.
- By applying a magnetic field to the plane with a **delta interaction** on a broken line, the discrete spectrum persists, at least for $\boldsymbol{\phi \lesssim \pi/8}$ $\boldsymbol{(\sim 23^o)}$