Amphibious Complex orbits and Dynamical Tunneling

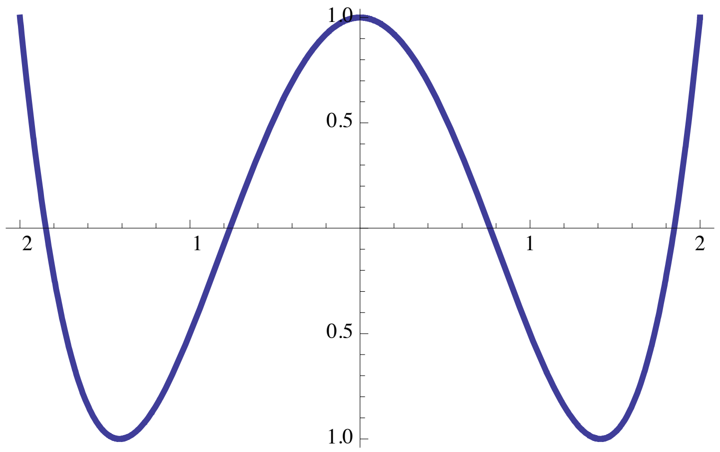
Akira Shudo (TMU)

in honor of Professor Petr **Š**eba's 60th Birthday

Hamiltonian systems in 1D and 2D

1D systems

$$H = \frac{p^2}{2} + V(q)$$



Classically, always completely integrable.

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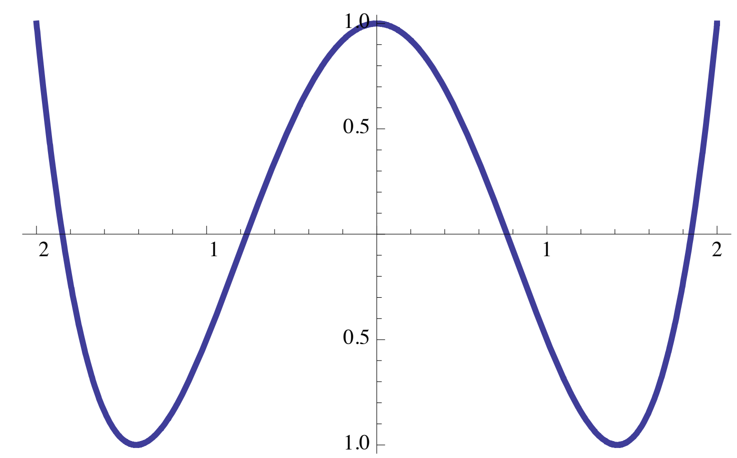
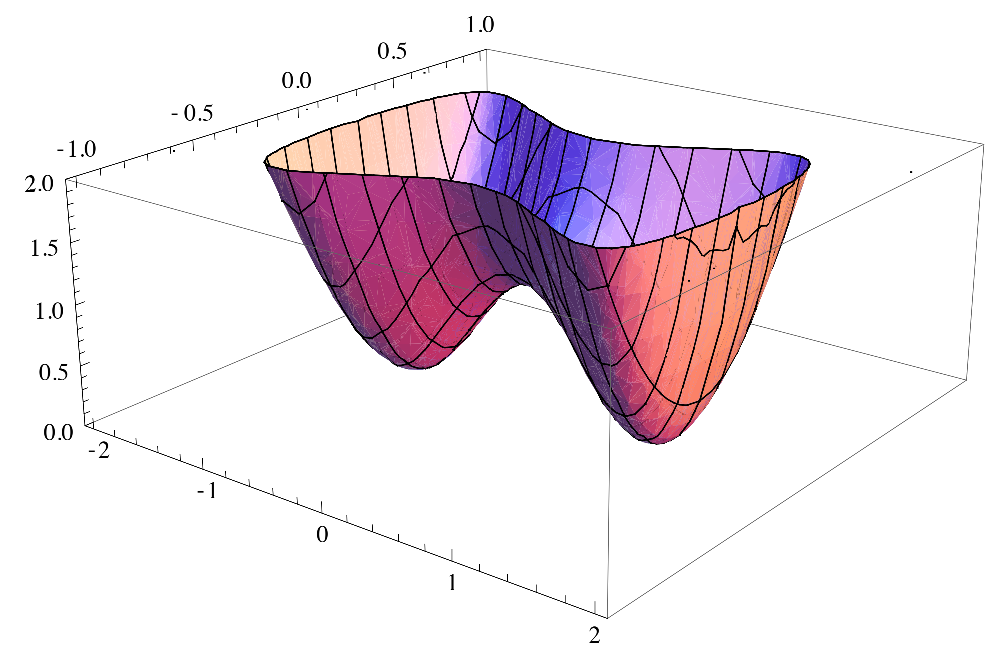


2D systems

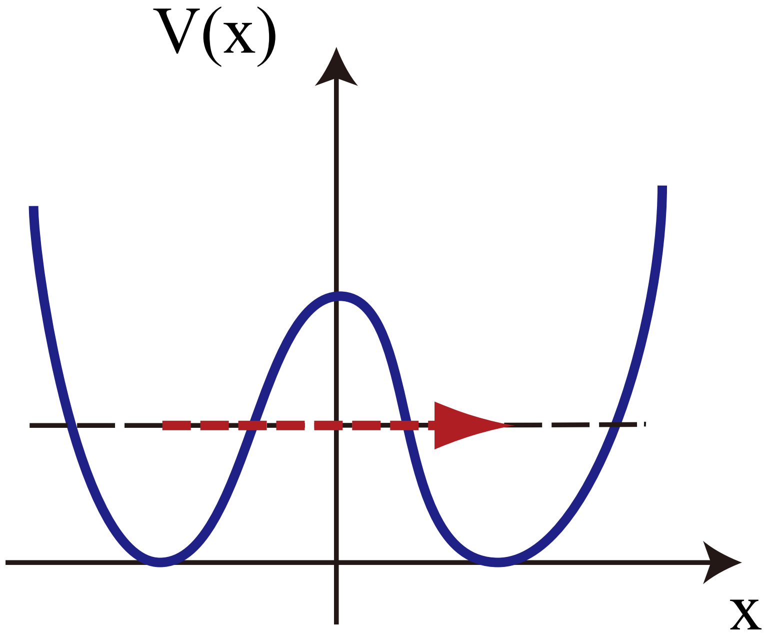
$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2} + V(q_1, q_2)$$

Classically, nonintegrable in general.



Quantum tunneling and complex paths

1D double well potential



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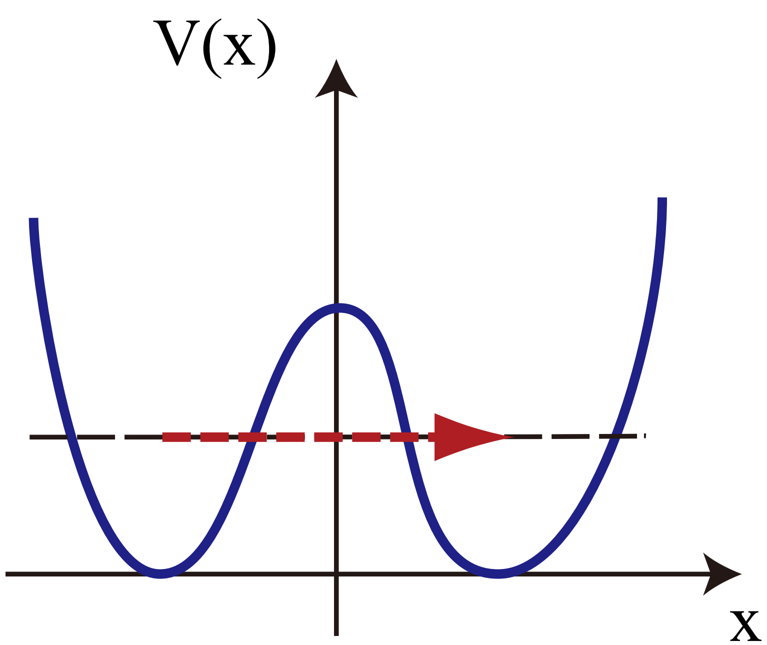
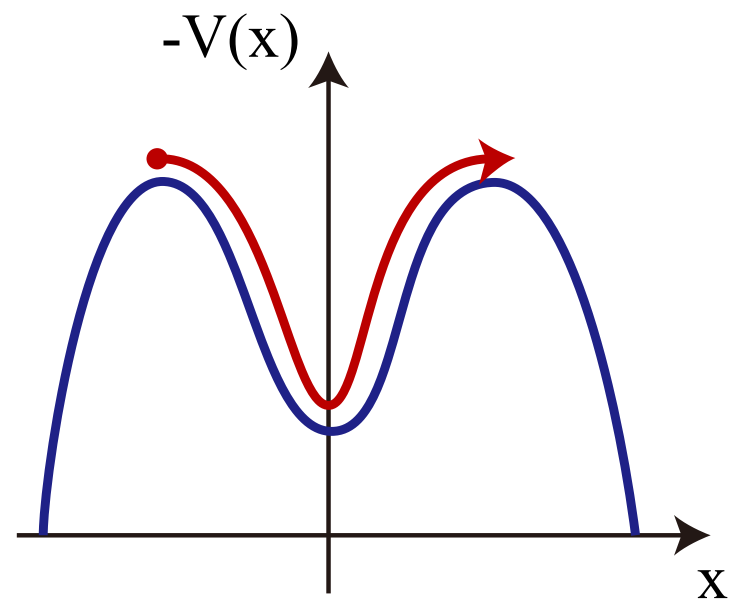
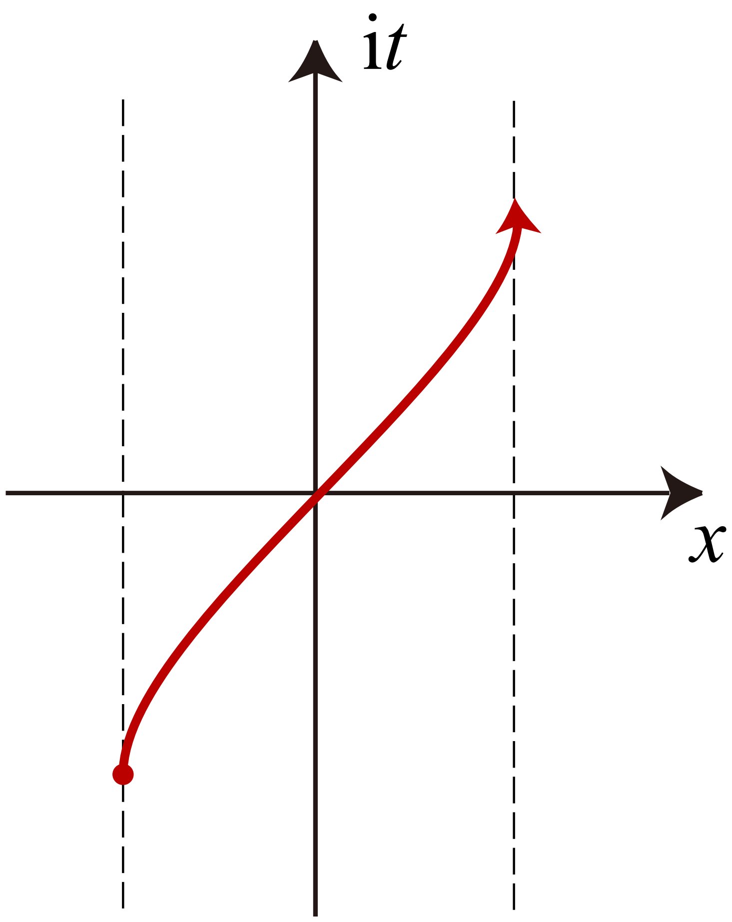


Instanton

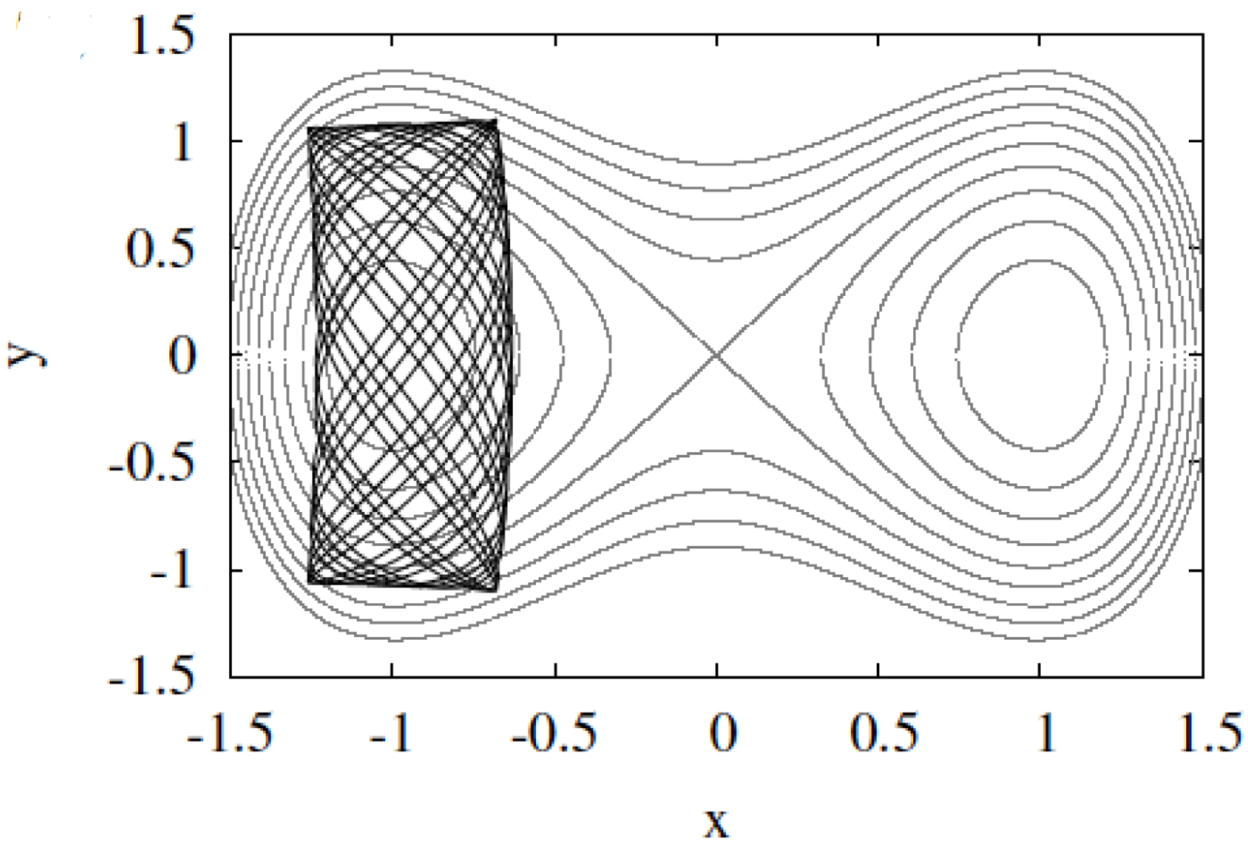




Quantum tunneling in multidimensions

2D double well potential

configuration space



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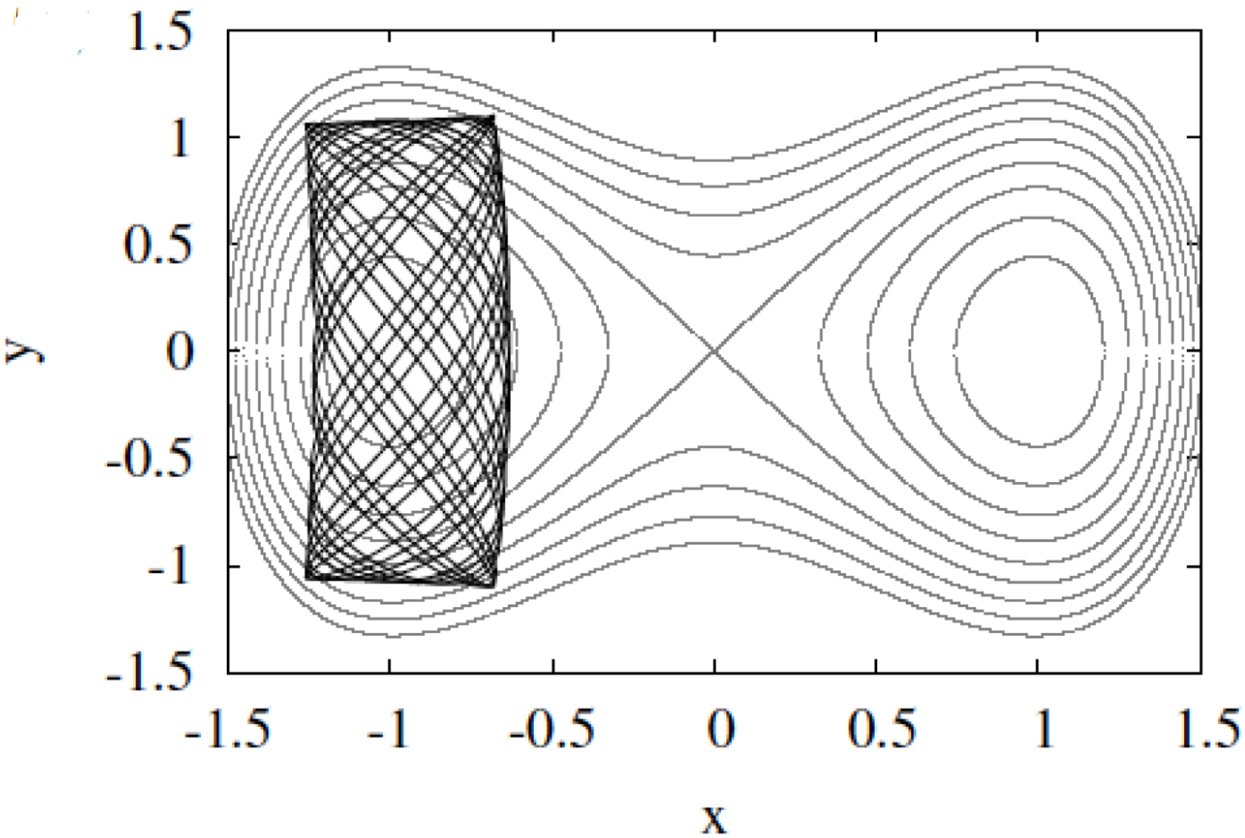
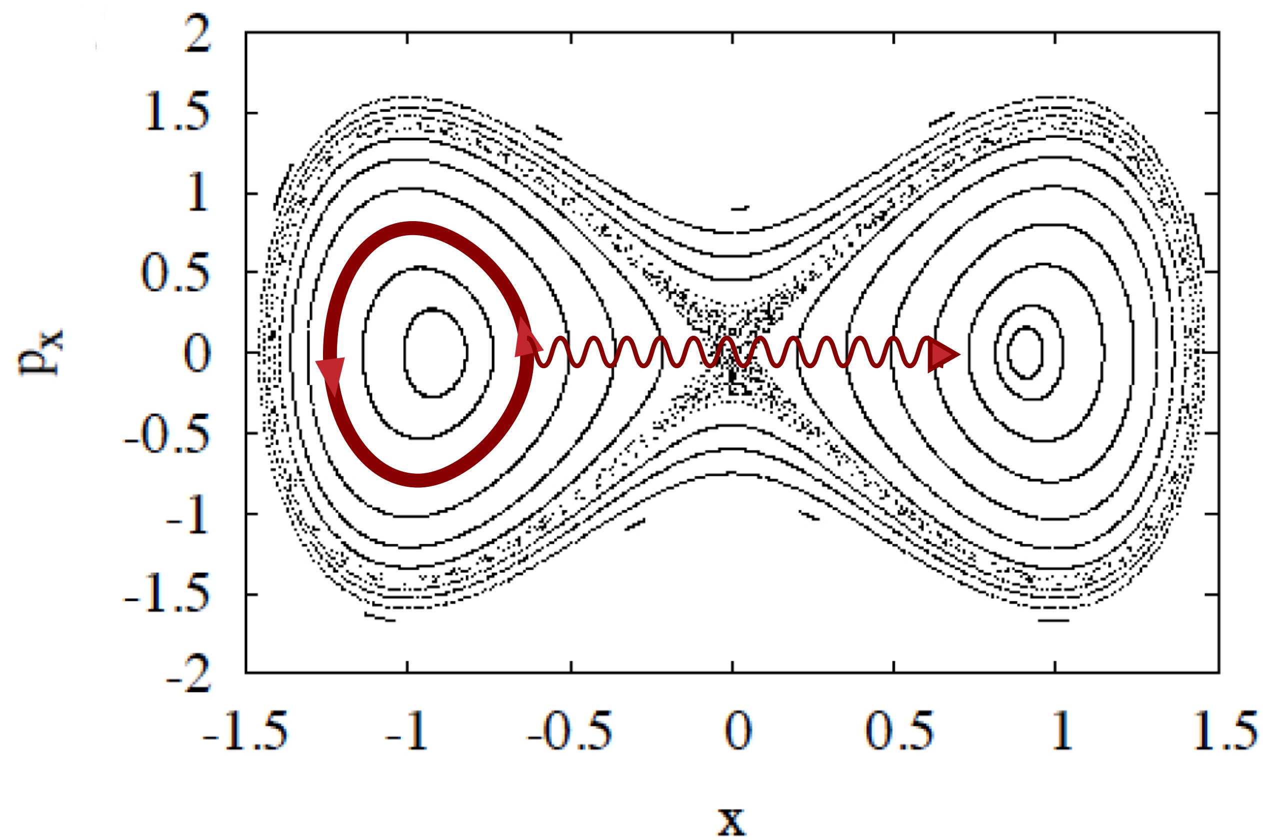


Poincaré section of phase space



- **Confinement occurs not only due to energy barriers, but also *dynamical barriers***
- **Chaos appears in phase space**

Area-preserving map and mixed phase space

Area-preserving map

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p - V'(q) \\ q + p' \end{pmatrix}$$

Area-preserving map and mixed phase space

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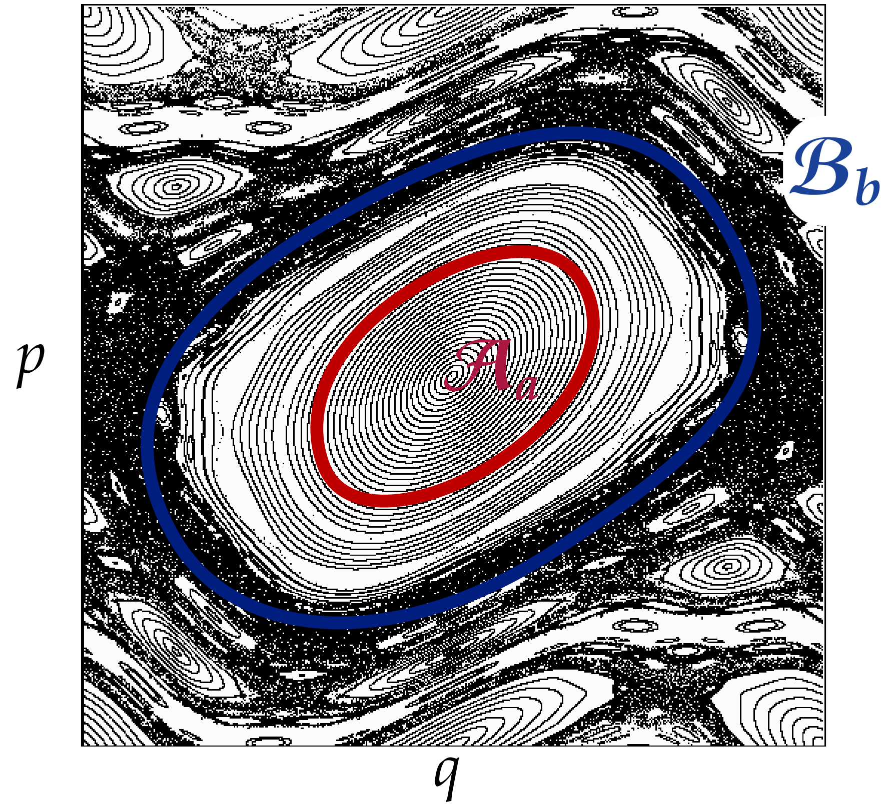
Forbidden process in classical dynamics

$\mathcal{A}_a \cap F^{-n}(\mathcal{B}_b) = \emptyset$ **for** $\forall n$**, if** $\mathcal{A}_a, \mathcal{B}_b (\in \mathbb{R})$ **are dynamically separated.**



Area-preserving map and mixed phase space

Quantum map

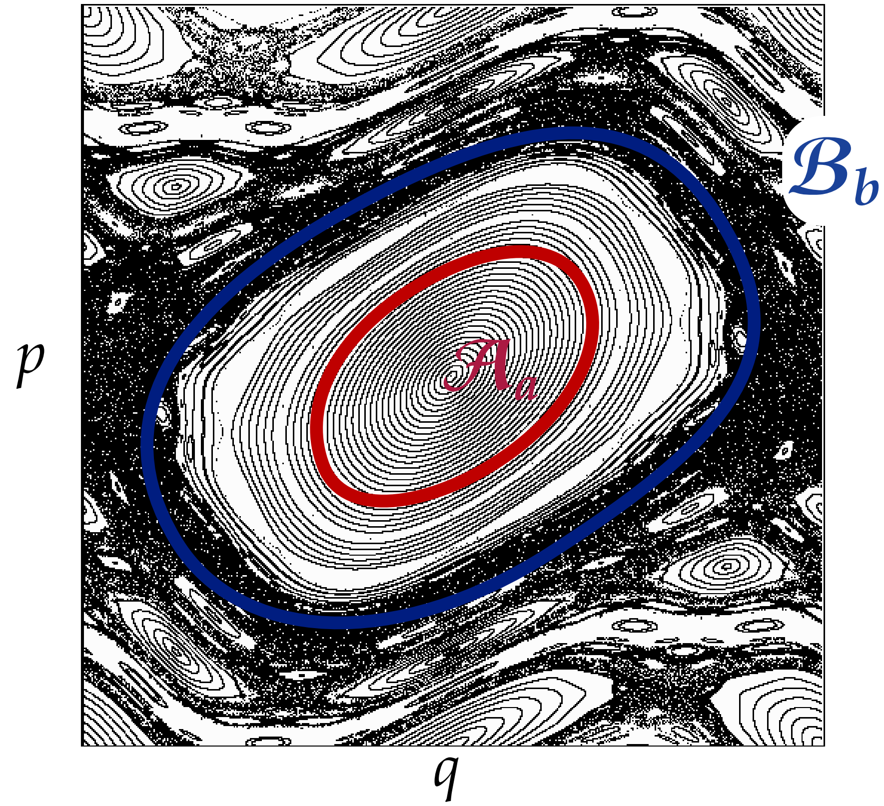
$$\hat{U} = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}T(\hat{p})}\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}V(\hat{q})}$$

Propagator

$$K(a,b) = \langle b|\hat{U}^n|a\rangle = \int\cdots\int \prod_j dq_j \prod_j dp_j \exp\left[\frac{i}{\hbar}S(\{q_j\},\{p_j\})\right]$$

Tunneling process in quantum dynamics

$K(a,b) \neq 0$ even if $\mathcal{A}_a, \mathcal{B}_b (\in \mathbb{R})$ are dynamically separated.



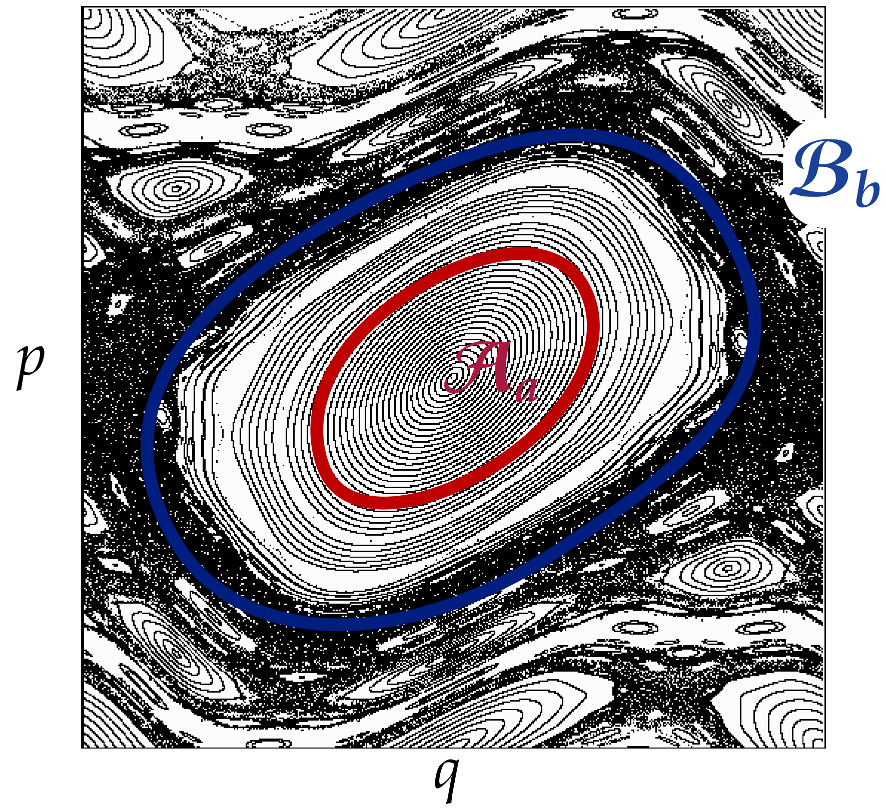
Area-preserving map and mixed phase space

Semiclassical approximation (Van-Vleck, Gutzwiller)

$$K^{sc}(a,b) = \sum_{\gamma} A_n^{(\gamma)}(a,b) \exp\left\{\frac{\mathrm{i}}{\hbar} S_n^{(\gamma)}(a,b)\right\}$$

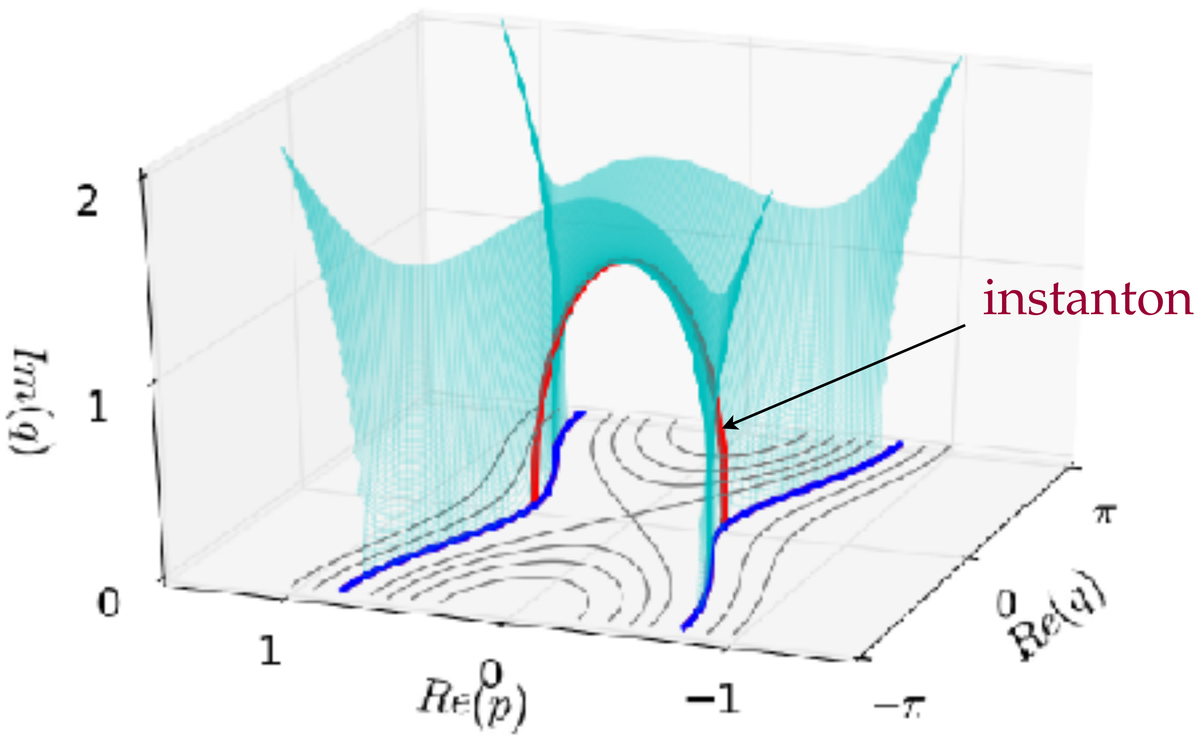
γ: classical orbits connecting a and b

If $F^n(\mathcal{A}) \cap \mathcal{B} = \emptyset$, then γ should be complex.



Complexified invariant manifold and natural boundaries

Completely integrable



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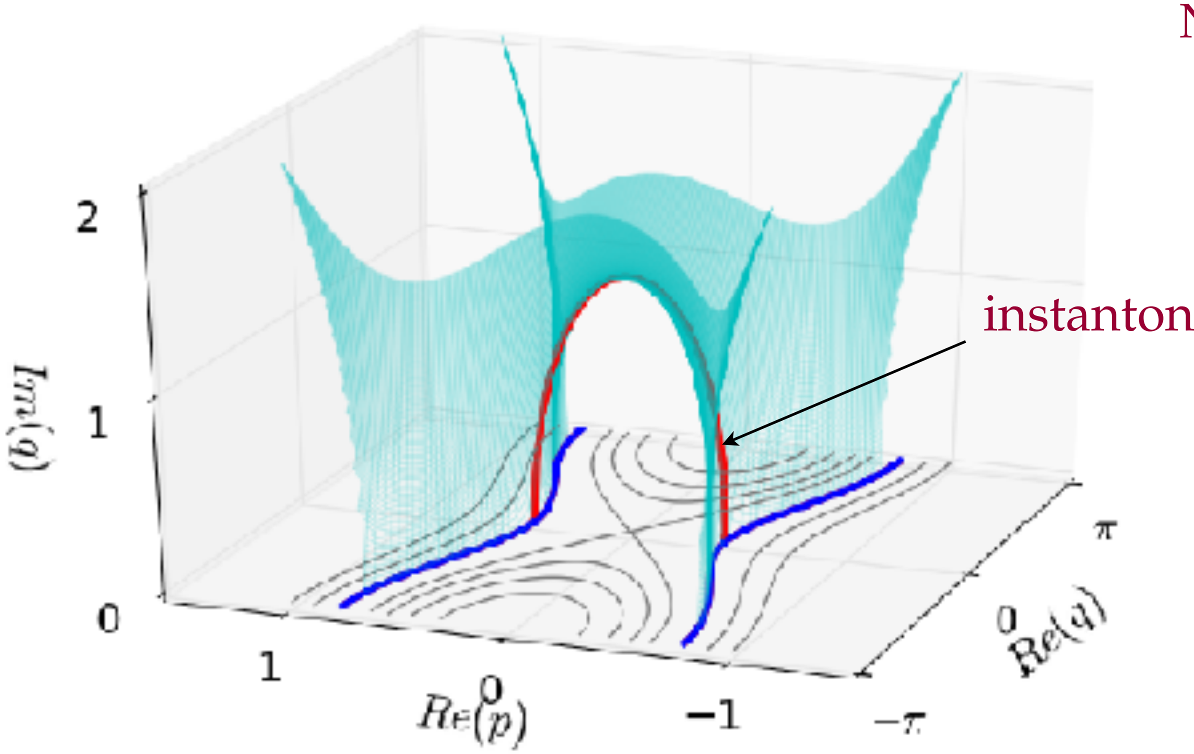
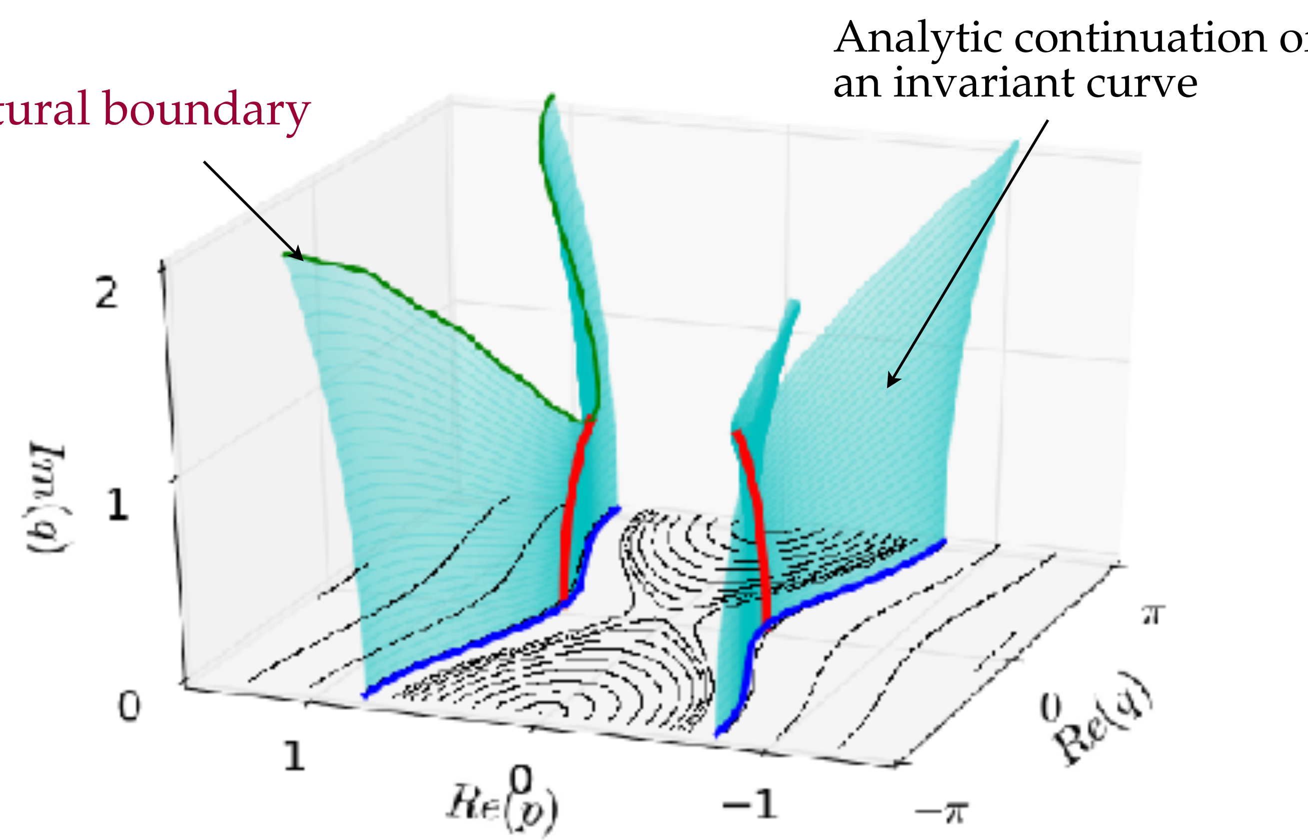


Nonintegrable



(Greene, Percival, Berretti, Marmi, $\cdots$)

How are disconnected regions connected ?

Complex dynamics in 1-dimensional maps

1-dimensional maps $F : \mathbb{C} \mapsto \mathbb{C}$

$$F : z' = F(z)$$

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Classify the orbits according to the behavior of $n \to \infty$

$$I = \{ z \in \mathbb{C} \mid \lim_{n\to\infty} F^n(z) = \infty \} \quad : \quad \text{Set of escaping points}$$

$$K = \{ z \in \mathbb{C} \mid \lim_{n\to\infty} F^n(z) \text{ is bounded} \} \quad : \quad \text{Filled Julia set}$$

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In particular

$J = \partial K$: **Julia set**

$F = \mathbb{C} - J$: **Fatou set**

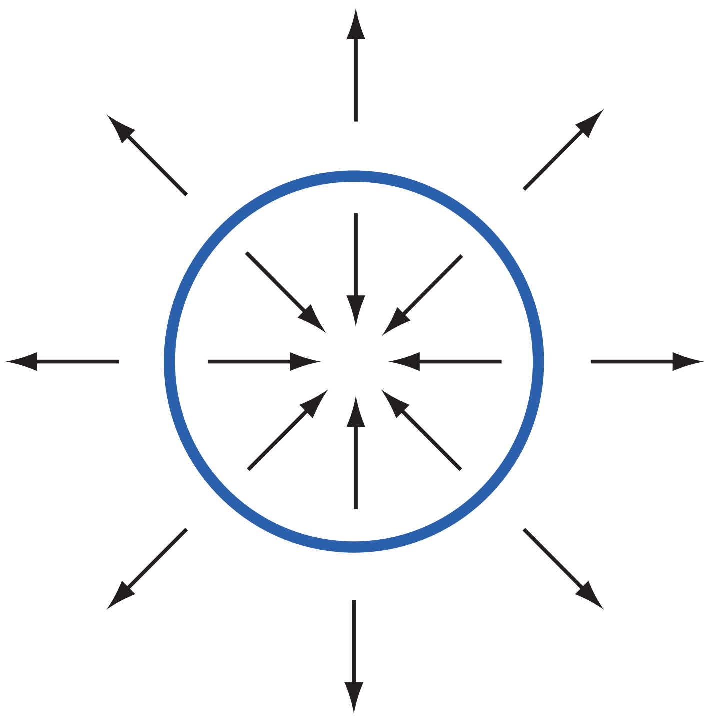
Julia set in 1-D dynamics

- $F(z) = z^2$

  $I = \{\, |z| > 1 \,\}, \quad K = \{\, |z| \leq 1 \,\}, \quad J = \{\, |z| = 1 \,\}$

  - $z = 0$ and $z = \infty$ are both attracting fixed points of F.
    The points $z \in I$ tend to ∞ and also the points $z \in K - J$ converge to $z = 0$ monotonically.

  - The orbits $z \in J$ are chaotic.
    Putting $z = e^{2\pi i\theta}$, then the map on J can be reduced to
    $\theta \mapsto 2\theta \pmod 1$.



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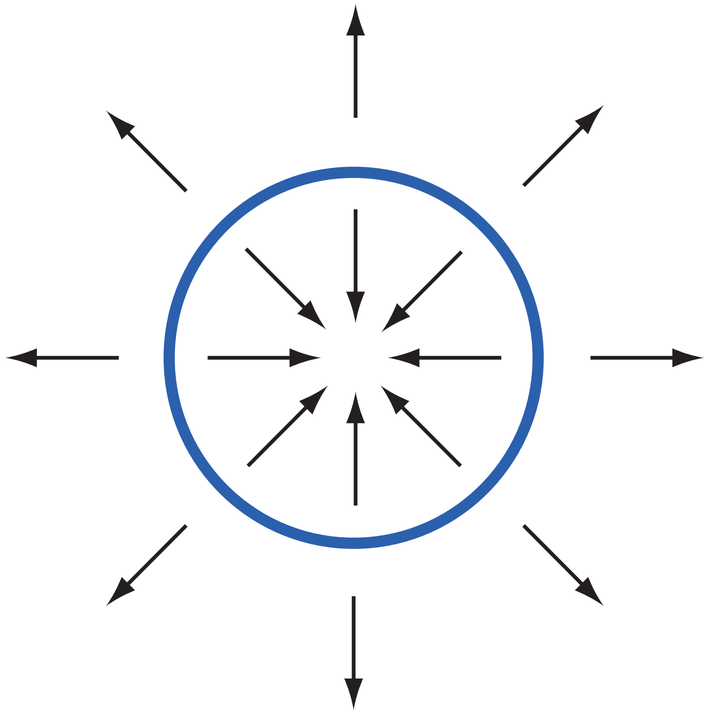
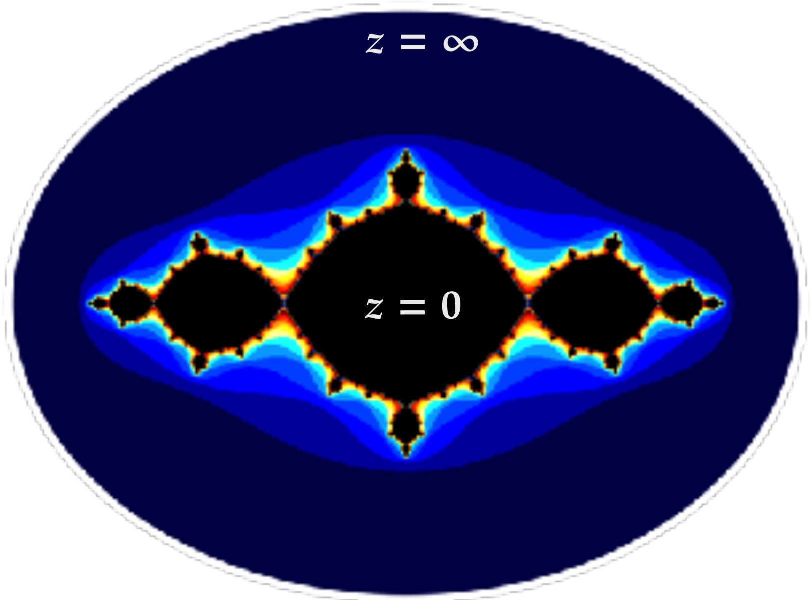


- $F(z) = z^2 + c$



Complex dynamics in 2-dimensional maps

2-dimensional maps $F : \mathbb{C}^2 \mapsto \mathbb{C}^2$

$$F : \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} f(z_1, z_2) \\ g(z_1, z_2) \end{pmatrix}$$

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Orbits are classified according to the behavior of $n \to \pm\infty$

$$I^{\pm} = \{ (z_1, z_2) \in \mathbb{C}^2 \mid \lim_{n\to\infty} F^{\pm n}(z_1, z_2) = \infty \}$$

$$K^{\pm} = \{ (z_1, z_2) \in \mathbb{C}^2 \mid \lim_{n\to\infty} F^{\pm n}(z_1, z_2) \text{ is bounded in } \mathbb{C}^2 \}$$

Complex dynamics in 2-dimensional maps

2-dimensional maps $F : \mathbb{C}^2 \mapsto \mathbb{C}^2$

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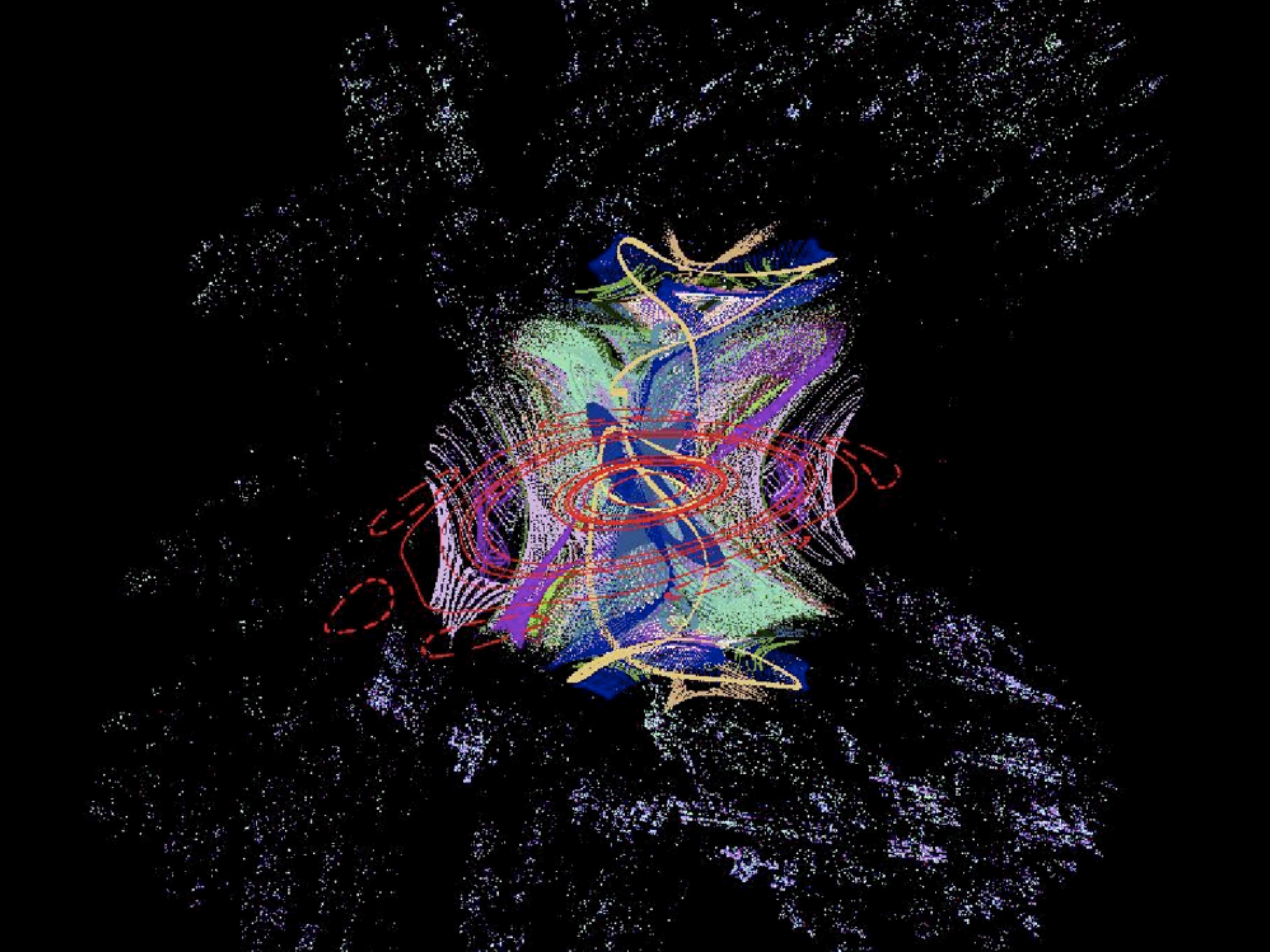
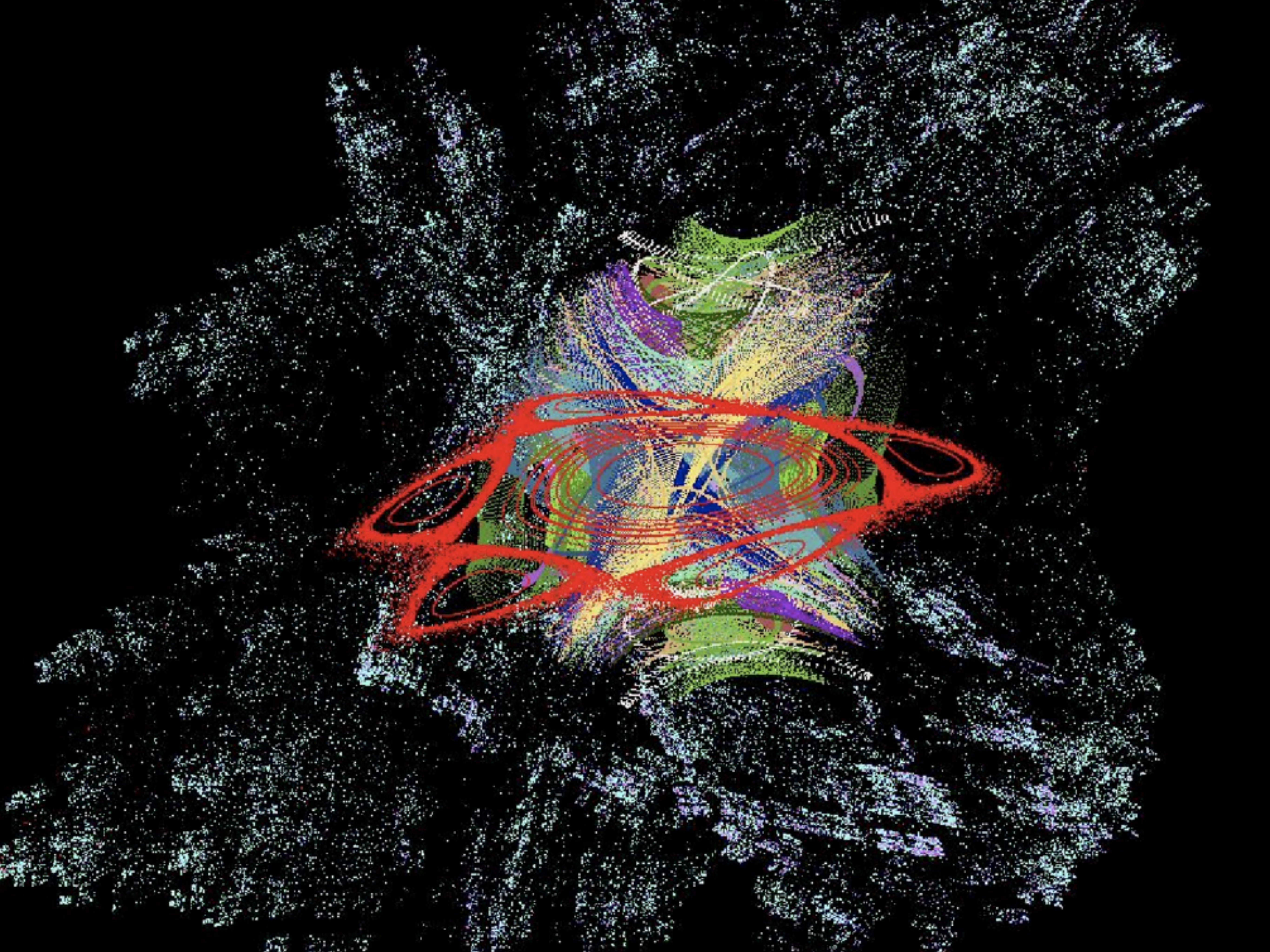
In particular

$K = K^+ \cap K^-$	:	filled Julia set
$J^\pm = \partial K^\pm$	:	forward (resp. backward) Julia set
$J = J^+ \cap J^-$	:	Julia set





Stable and unstable manifold theorem

$$F : \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p + V'(q) \\ q + p' \end{pmatrix} \qquad (\, V(q) : \text{polynomial} \,)$$

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Theorem (Bedford-Smillie 1991) *For any unstable periodic orbits* $\mathbf{p}$,

$$\overline{W^s(\mathbf{p})} = J^+ \quad \textit{and} \quad \overline{W^u(\mathbf{p})} = J^-$$

where $W^s(\mathbf{p})$ (*resp.* $W^u(\mathbf{p})$) *denotes stable* (*resp. unstable*) *manifold for* $\mathbf{p}$ *and*

$J^\pm = \partial K^\pm$ *is called the forward* (*backward*) *Julia set.*

Here, $K^\pm = \{\, (p, q) \in \mathbb{C}^2 \mid \|F^n(p, q)\| \text{ is bounded } (n \to \pm\infty) \,\}$

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Note :

1. This theorem holds even in the system with mixed phase space.
2. $W^s(\mathbf{p})$ and $W^u(\mathbf{p})$ are both locally 1-dimensional complex (= 2-dimensional real) manifold in $\mathbb{C}^2$.

Amphibious complex orbits and flooding

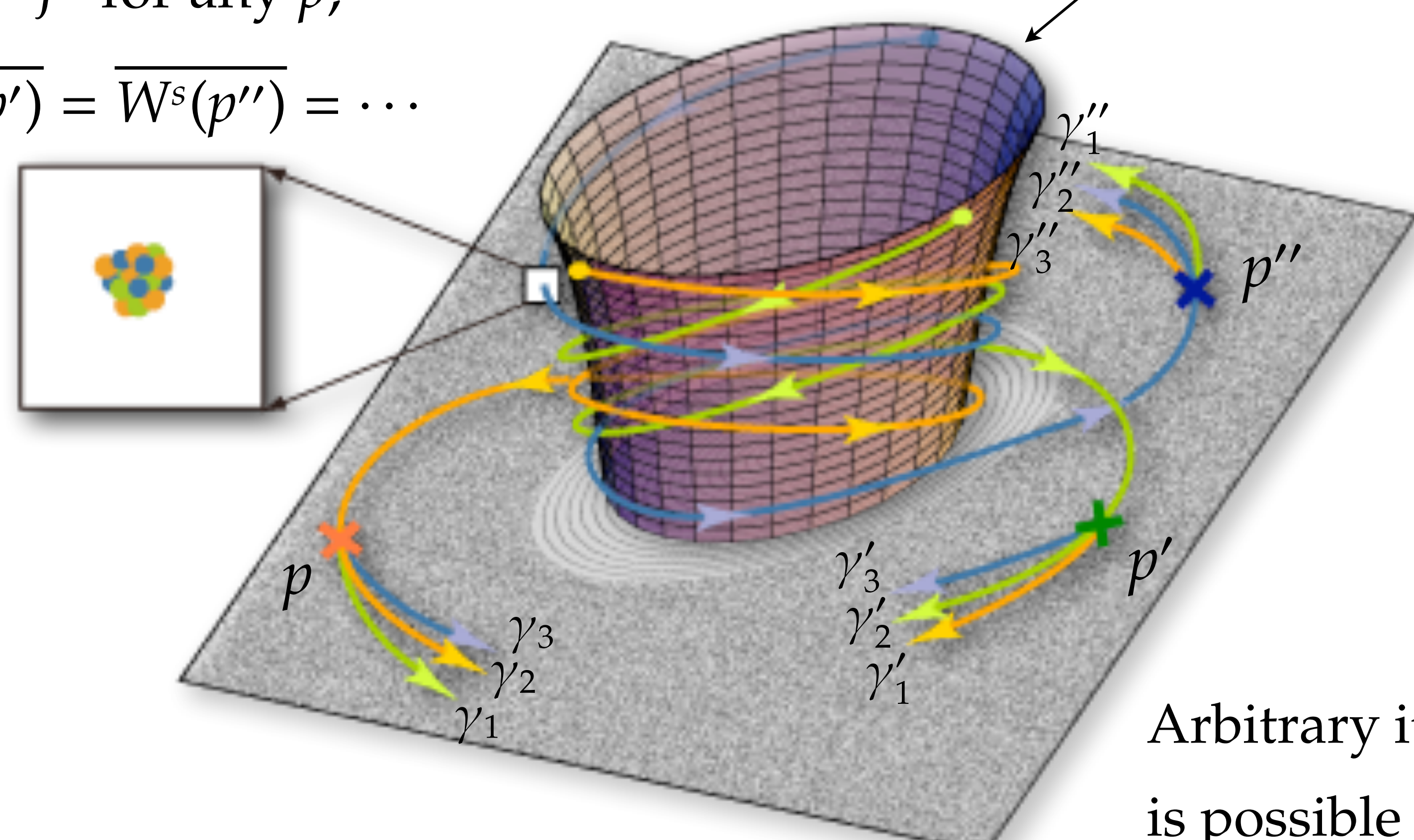
- **The orbits behave as regular orbits when they stay in the regular region, while they become chaotic after reaching the chaotic sea.**

- **Exponentially many complex orbits connect regular chaotic states and flood the entire region.**

Since $\overline{W^s(p)} = J^+$ for any p,

$\overline{W^s(p)} = \overline{W^s(p')} = \overline{W^s(p'')} = \cdots$

initial state a



Arbitrary itineracy $p \to p' \to p'' \to \cdots$ is possible after reaching $\mathbb{R}^2$.

Tunneling orbits and Julia sets

Semiclassical sum

$$K^{sc}(a,b) = \sum_{\gamma} A_n^{(\gamma)}(a,b) \exp\left\{\frac{\mathrm{i}}{\hbar} S_n^{(\gamma)}(a,b)\right\}$$

Theorem (AS, Y. Ishii and K.S. Ikeda) **For polynomial maps F,**

(i) If F is hyperbolic and $h_{\mathrm{top}}(F|_{\mathbb{R}^2}) = \log 2$, then $C = J^+$

(ii) If F is hyperbolic and $h_{\mathrm{top}}(F|_{\mathbb{R}^2}) > 0$, then $\overline{C} = J^+$

(iii) If $h_{\mathrm{top}}(F|_{\mathbb{R}^2}) > 0$, then $J^+ \subset \overline{C} \subset K^+$

Here $h_{\mathrm{top}}(P|_{\mathbb{R}^2})$ is topological entropy confined on $\mathbb{R}^2$, and semiclassically contributing complex orbits are introduced as

$$C \equiv \{ (q,p) \in \mathcal{M}_\infty \mid \mathrm{Im}\, S_n(q,p) \text{ converges absolutely at } (q,p) \}$$

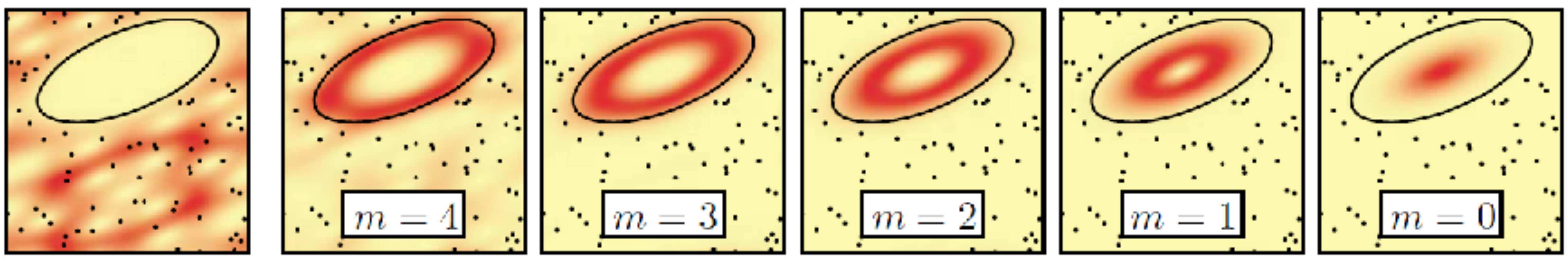
(Proof) apply the convergent theory of current (Bedford-Smillie)

Violation of semiclassical eigenfunction hypothesis ?

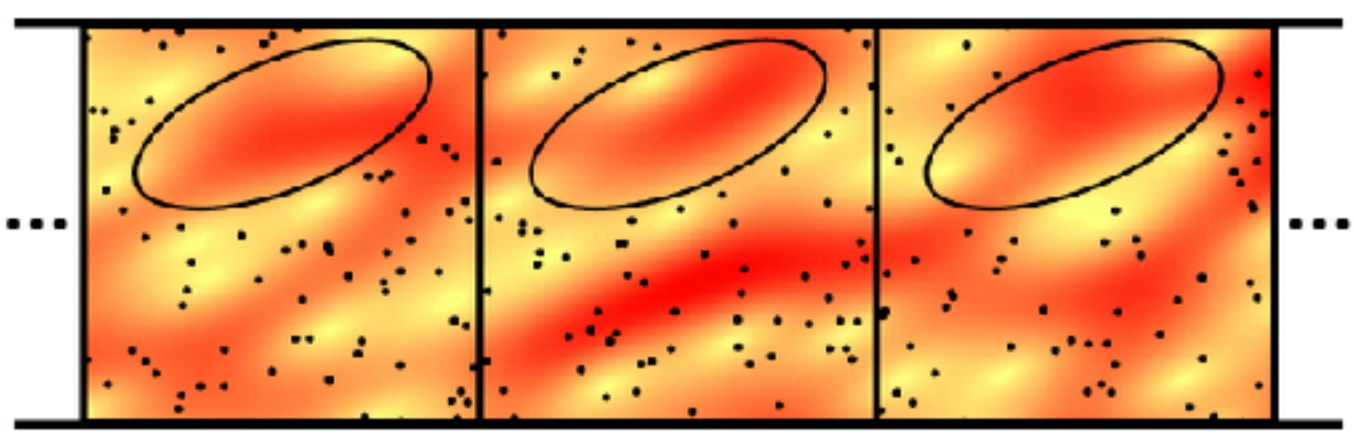
"Eigenstates ignores regular and chaotic phase space structures"
L.Hufnagel, R.Ketzmerick, M.F.Otto and H.Schanz, PRL, 89, 154101-1 (2002)

"Flooding of chaotic eigenstates into regular phase space islands"
A.Bäcker, R.Ketzmerick, A.G.Monasta, PRL, 94, 045102-1 (2005)

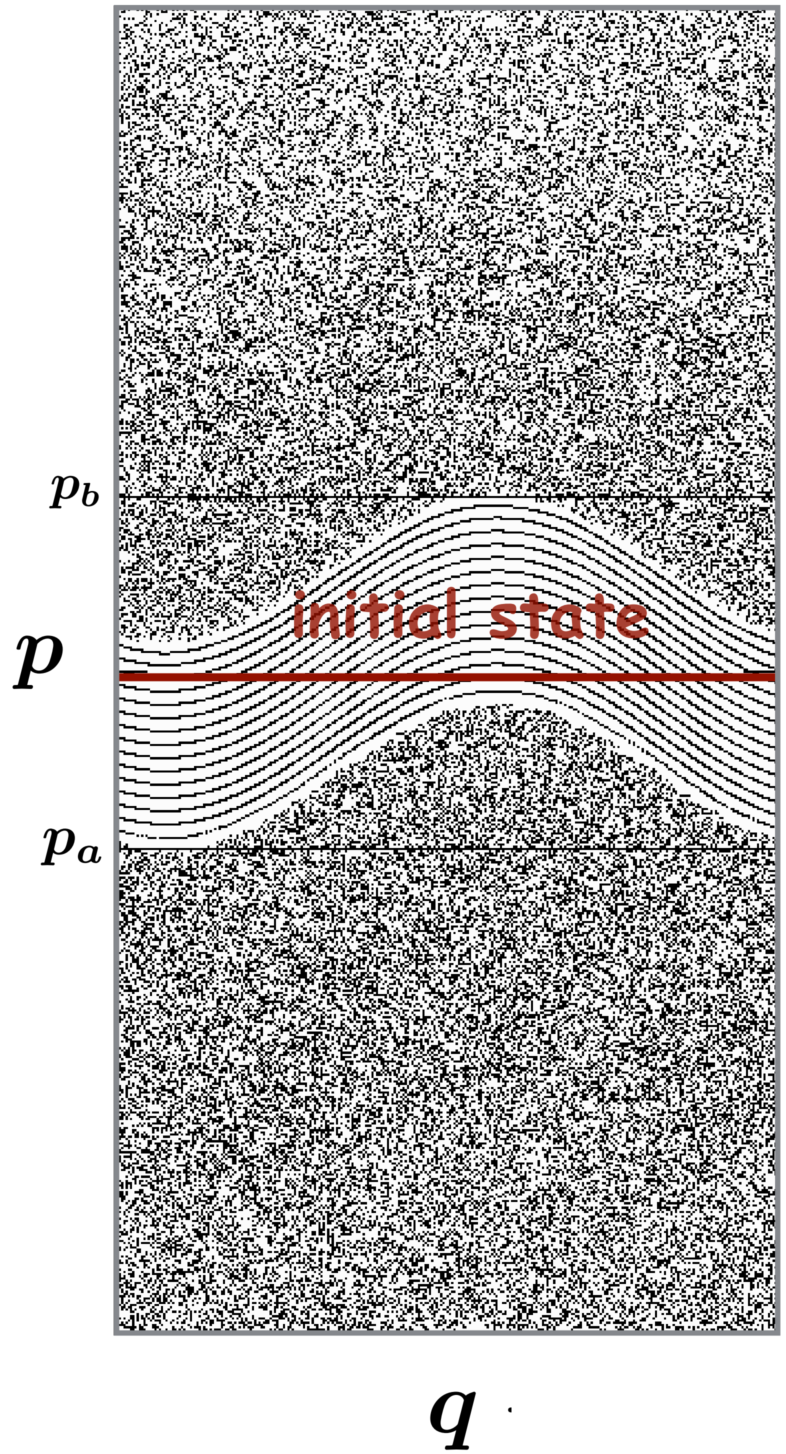
Localized states (either on torus or chaotic region)



Amphibious states (extended over torus and chaotic region)

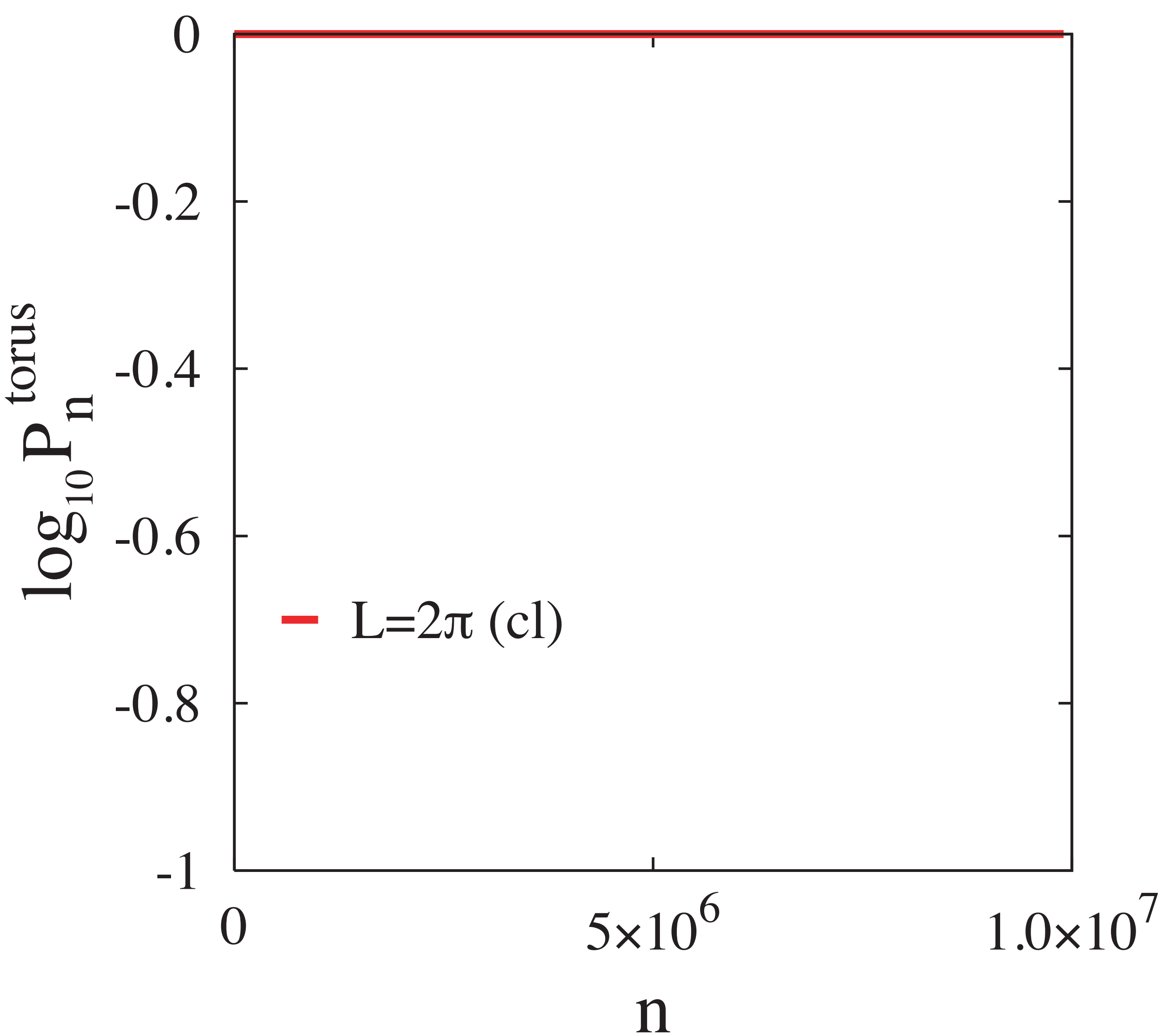


Flooding induced by destructing coherence in the chaotic region



Survival probability in the regular region (classical)

$$P_n^{\text{torus}} = \int_{p_a}^{p_b} P_n(p)dp$$



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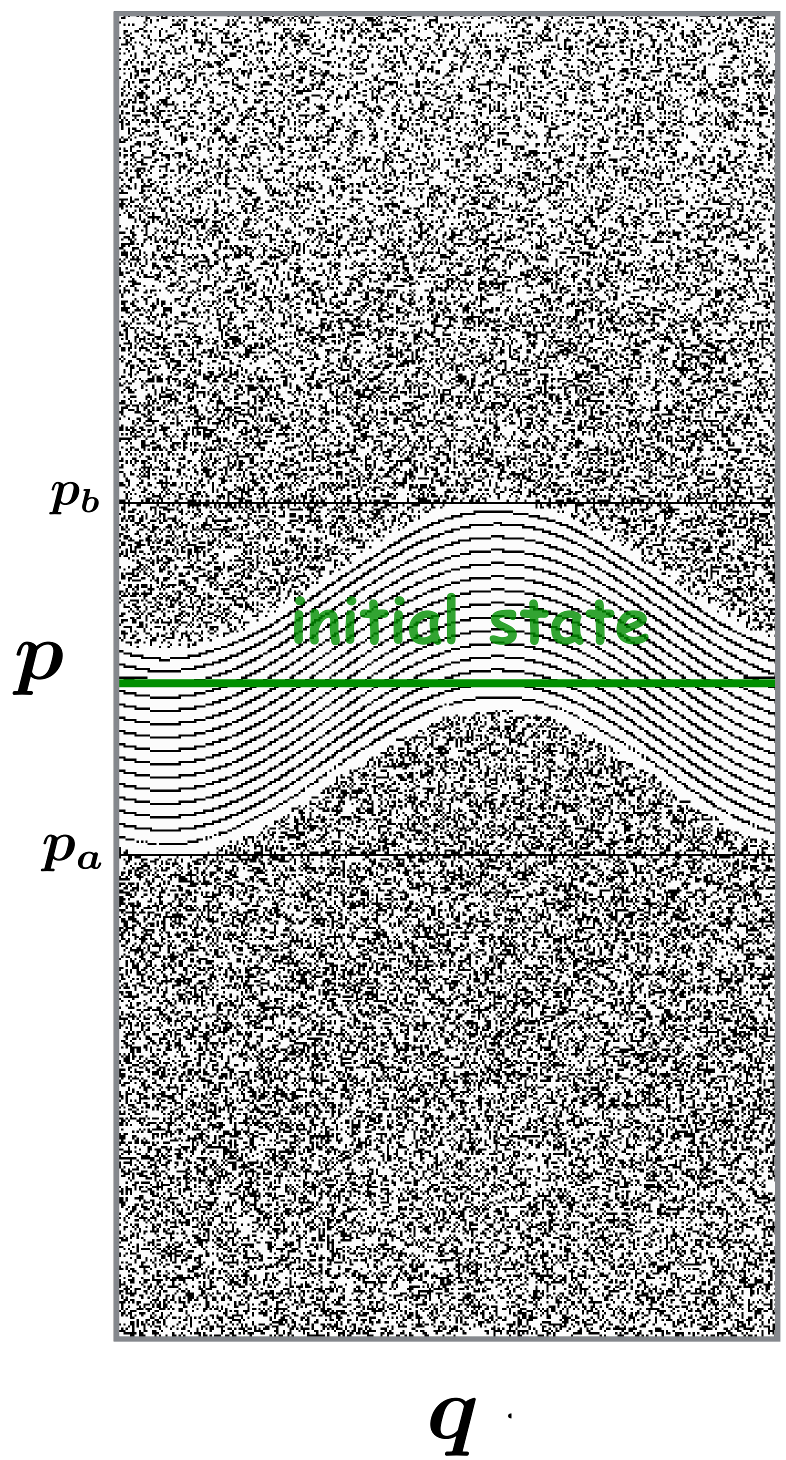
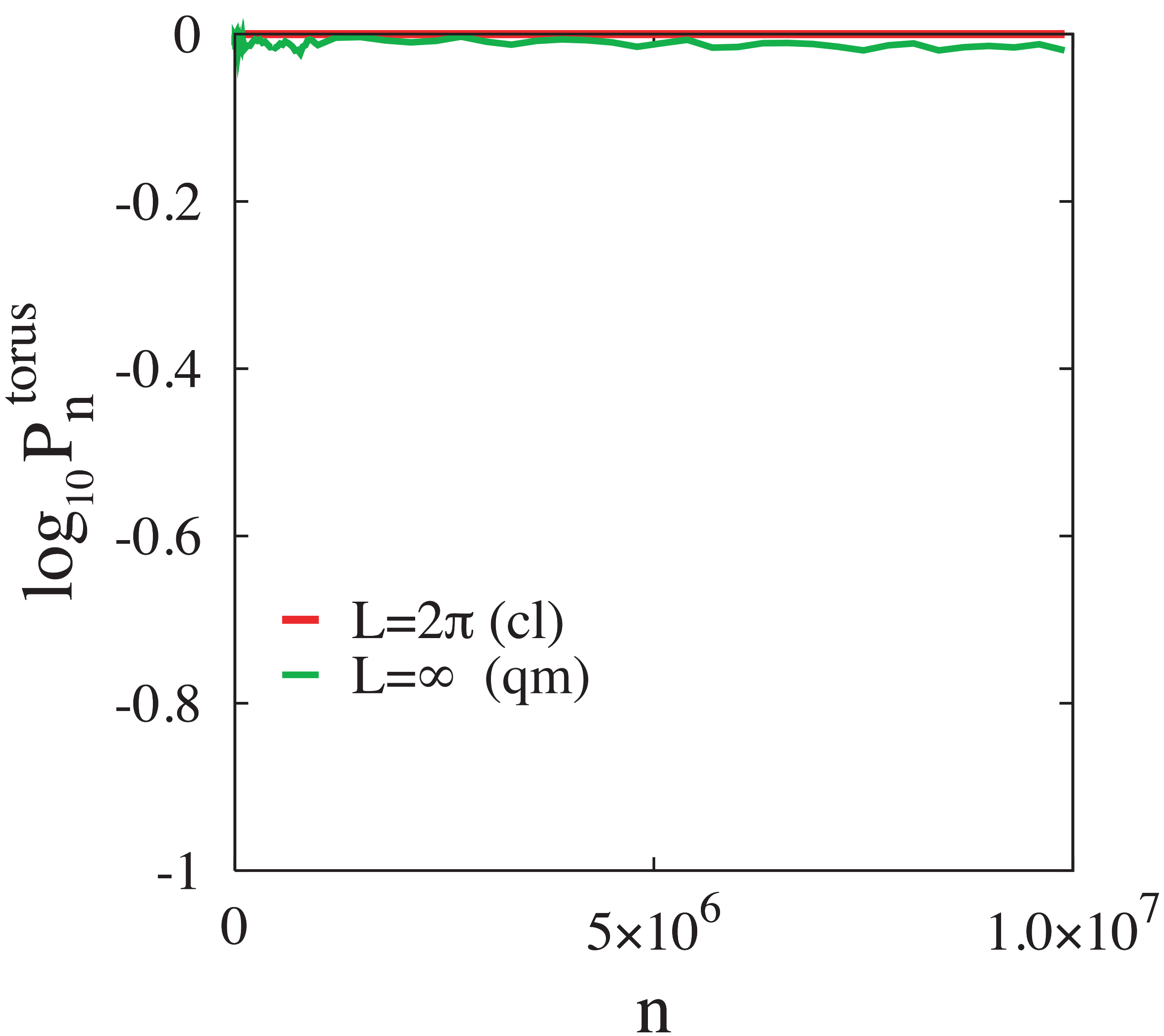


Survival probability in the regular region (quantum)

$$P_n^{\text{torus}} = \int_{p_a}^{p_b} |\psi_n(p)|^2 dp$$



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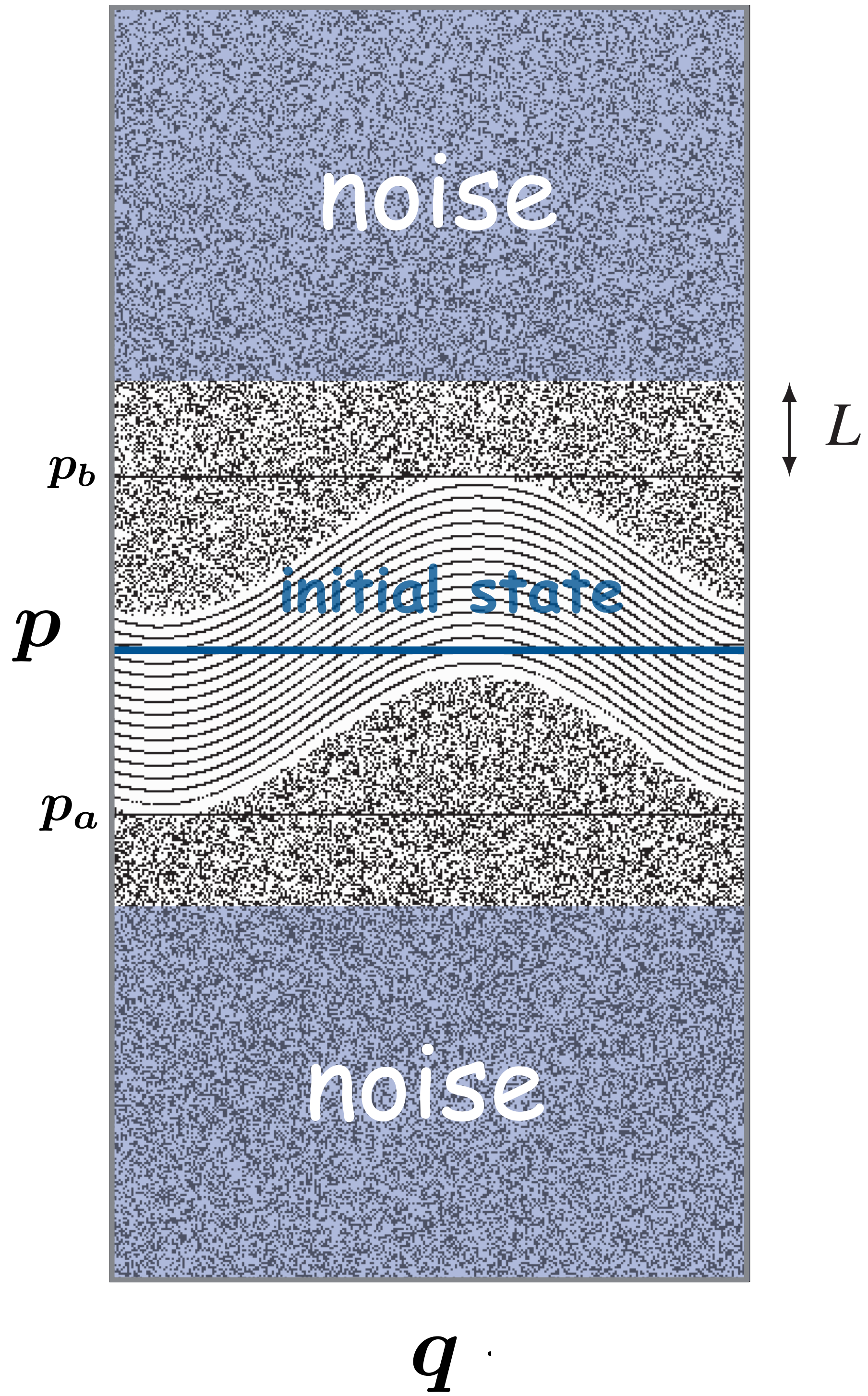

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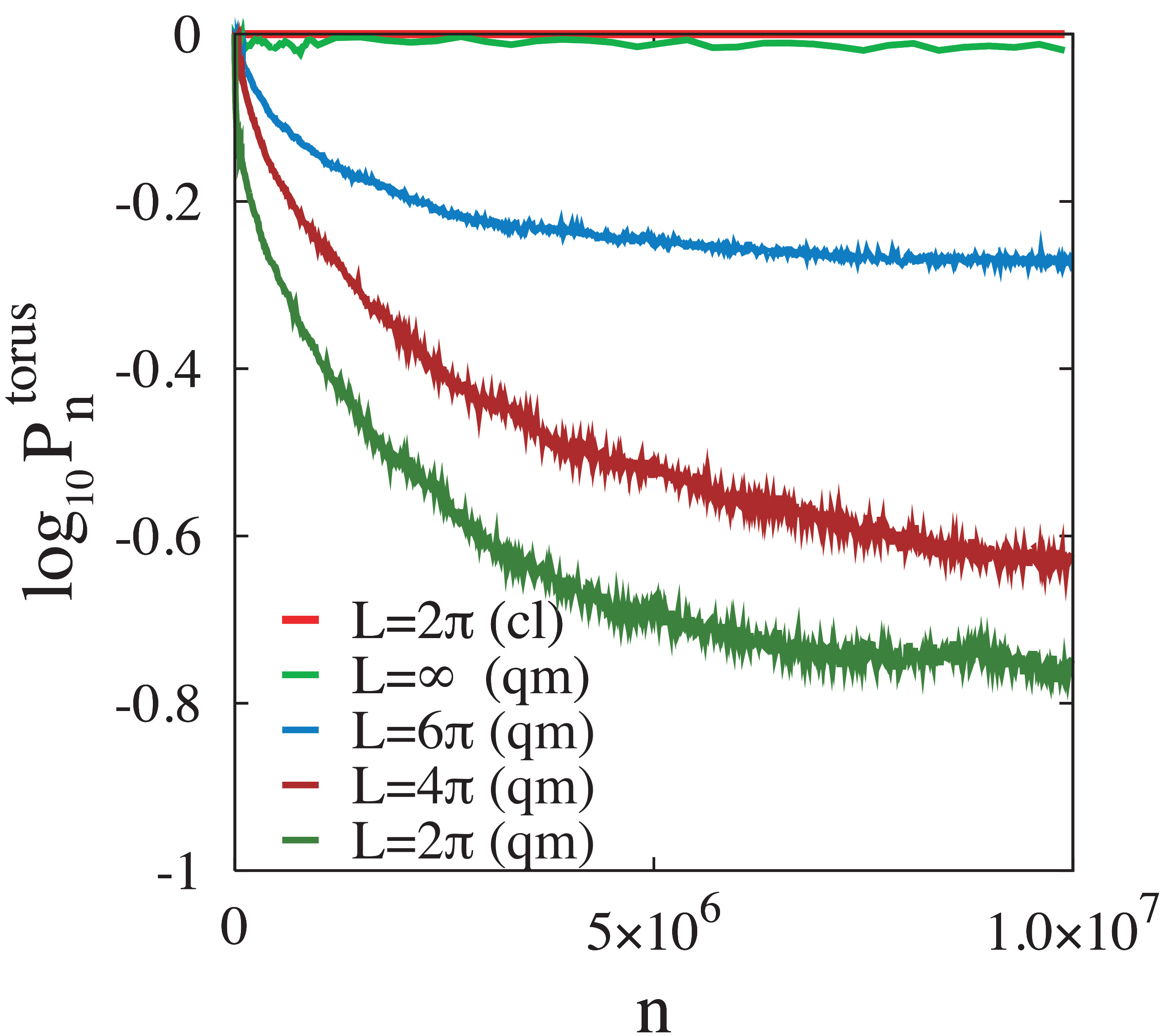

Summary

- **Instanton disappears due to the presence of *natural boundaries* in generic multidimensional systems.**

- **Classically disconnected regions are connected via orbits in the Julia set.**

- **The orbits in the Julia set are ergodic and have *amphibious* nature.**

- **Exponentially many orbits potentially exist behind the tunneling process from regular to chaotic regions but they are blocked by dynamical localization in the chaotic region.**

- **Flooding occurs if quantum interference effects are suppressed either by adding noise or by coupling the systems to invoke the Anderson transition in the chaotic region.**