Spectral and Resonance Properties
of the Smilansky model

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Outline:
1. Physical motivation
2. The model
3. Some numerical results

Irreversible quantum graphs

- Quantum graph: a system consisting of a metric graph Γ and a self-adjoint operator acting on $L^2(\Gamma)$ (usually the Laplacian)
- quantum graphs emulate many properties of quantum systems; many physical systems are irreversible $\Rightarrow$ it is desirable to include it in the theory
- to introduce irreversibility to a quantum system: by coupling it to a 'bath' (it consists of 'irrelevant' degrees of freedom)
- the total system is described by $H = H^{system} + H^{bath} + V$
- H^{system} and H^{bath} are defined in respective Hilbert spaces
- irreversibility is generally associated with the bath having a continuous spectrum
- the most popular is the harmonic bath: a continuous set of harmonic oscillators linearly coupled to the system (the decay width introduced by the bath can be computed explicitly)
- The model: a graph linearly coupled to a harmonic bath

Informal setting of the problem

Let Γ be a metric star graph with m bonds, all emanating from a common vertex (the root of Γ).

The differential expression for H acts on functions $\psi(x, y)$ where x runs over Γ (compact or non-compact) and y runs over the real line $\mathbb{R}$.

The differential expression is a combination of the Laplacian in x and HO in y; it does not involve λ.

$$\psi(x, y) = \sum_n u_n(x) f_n(y)$$

The components $u_n(x)$ are coupled by a system of matching conditions at the root of Γ. The coupling parameter λ appears in these conditions.

If $\lambda = 0$, $\sigma(H_0)$ is discrete for compact Γ and $\sigma(H_0) = (1/2, \infty)$ otherwise.

If $\lambda > 0$ and small, $\sigma(H_\lambda)$ is similar to $\sigma(H_0)$. However, for λ large, the operator is unbounded from below and $(-\infty, 1/2]$ belongs to its continuous spectrum, with no embedded eigenvalues (even if the Γ is compact).

So, there is a point λ_0 (depending on the structure of Γ) that separates these two ranges of λ.

Less informal setting of the problem

Let $u(x)$ be a function on Γ, $u^{(j)}(x)$ the restriction to the jth bond. The metric in the Sobolev space $\mathrm{H}^1(\Gamma)$ is

$$\int_\Gamma (|u'|^2 + |u|^2)dx := \sum_{j=1}^{m} \int_0^{B_j} (|(u^{(j)})'|^2 + |u^{(j)}|^2)dx$$

We are interested in the differential operator in $\mathrm{L}^2(\Gamma \times \mathbb{R})$:

$$H_\lambda = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right)$$

with the boundary conditions
$[u'(root)] := (u^{(1)})'(root) + \cdots + (u^{(m)})'(root)$

$$\begin{aligned} &\psi|_{\partial\Gamma\times\mathbb{R}} = 0 \\ &[\psi_x(root, y)] = \lambda y \psi(root, y), \qquad \forall y \in \mathbb{R} \end{aligned}$$

$$h_\lambda[\psi] = \int_{\Gamma\times\mathbb{R}} (|\psi_x'|^2 + \frac{1}{2}(|\psi_y'|^2 + y^2|\psi|^2))dxdy + \lambda \int_{\mathbb{R}} y|\psi(root, y)|^2 dy$$

allows us to view H_λ as the Schrödinger operator

$$H_\lambda = -\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \lambda y \delta(x)$$

The model

Let $m = 2$

$$\left[-\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left(-\frac{\partial^2}{\partial y^2} + y^2\right) + \lambda y \delta(x)\right]\psi(x,y) = E\psi(x,y)$$

The boundary condition at the vertex is replaced by a term which can be interpreted as a potential, linear in y and strongly localized in x.

Particle moving in 2D space and acted upon by $\lambda y \delta(x)$: it forms a valley in x-direction and a positive ridge ($\lambda > 0$ at the vicinity of $x = 0 \Rightarrow y > 0$ domain is inaccessible to the particle, while $y < 0$ may attract it.

In 2D case, any shallow attractive potential can support at least one bound state $\Rightarrow$ we may expect discrete spectrum.

In the *subcritical regime*, $0 < \lambda < \sqrt{2}$, the operator is positive, its spectrum is purely continuous above $1/2$ and has a discrete component in $(0, 1/2)$, while in the *supercritical regime*, $|\lambda| > \sqrt{2}$, the particle can escape to infinity along the singular "channel" in the y direction, and consequently, the spectrum covers the whole real line being absolutely continuous.

Exner, Barseghyan (2014): $y\delta(x) \to y^2 V(xy)\chi_{|x|\leq a}(x)$

It dates back to Znojil (1998).

Numerical treatment

Ansatz: $\psi(x,y) = \sum_{n=0}^{\infty} c_n e^{-\kappa_n x} f_n(y)$ with

$$\kappa_n := \sqrt{n + \frac{1}{2} - E} \qquad f_n(y) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-y^2/2} H_n(y)$$

into (the odd part is trivial, the even on $L^2(\mathbb{R} \times (0,\infty))$

$$\psi_x(0+,y) = \lambda y \psi(0+,y)/2$$

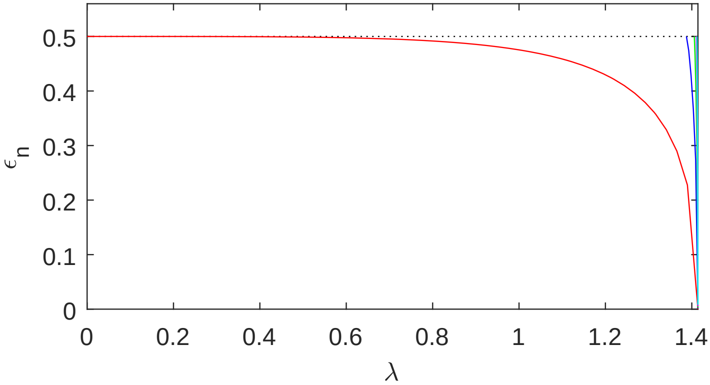
leads to $B_\lambda c = 0$, where c is the coefficient vector and B_λ is the operator in ℓ^2:

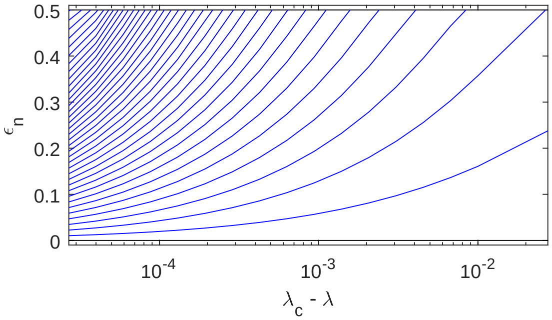
$$(B_\lambda)_{m,n} = \kappa_n \delta_{m,n} + \frac{1}{2}\lambda(f_m, y f_n)$$

This matrix is tridiagonal because

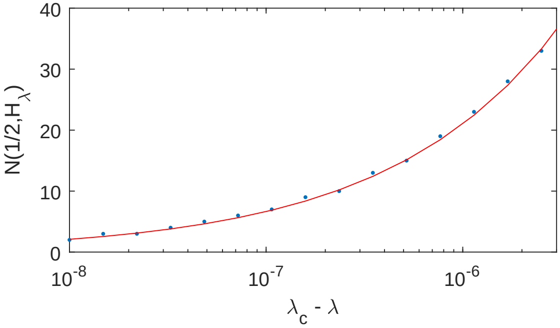
$$(f_m, y f_n) = \frac{1}{\sqrt{2}}(\sqrt{n+1}\delta_{m,n} + \sqrt{n}\delta_{m,n-1})$$

Some numerical results

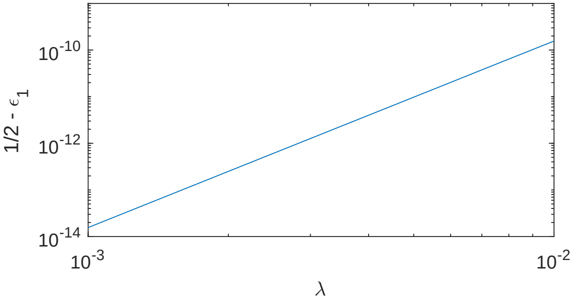




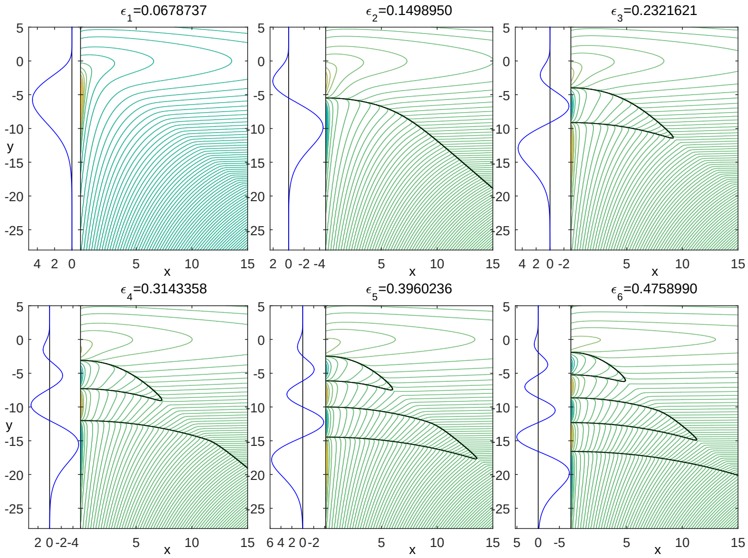
0.5
0.4
0.3
0.2
0.1
0
ϵ_n
10^{-4}
10^{-3}
10^{-2}
$\lambda_c - \lambda$



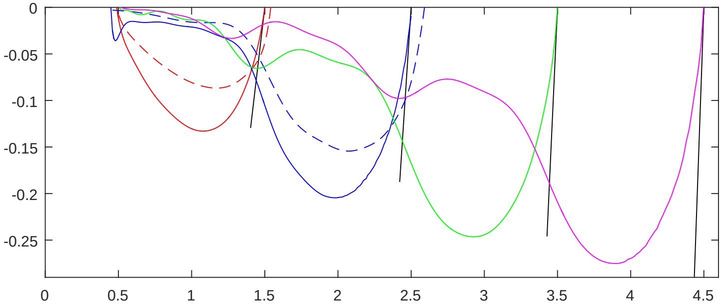
Solomyak: $N(\frac{1}{2}, H_\lambda) = \frac{1}{4}\sqrt{\frac{1}{\sqrt{2}(\sqrt{2}-\lambda)}}$



$$\epsilon_1(\lambda) = \frac{1}{2} - \frac{\lambda^4}{64} + \mathcal{O}(\lambda^5)$$



ϵ_1=0.0678737
ϵ_2=0.1498950
ϵ_3=0.2321621
ϵ_4=0.3143358
ϵ_5=0.3960236
ϵ_6=0.4758990
x
y



$$\rho_m(\lambda) = m + \frac{1}{2} - \frac{\lambda^4}{64}(2m + 1 + 2im(m+1)) + \mathcal{O}(\lambda^5)$$