Non-Hermitian Heisenberg picture

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M. Znojil, NPI Řež

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news on quantum unitary evolution in non-Hermitian representation in

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Schroedinger picture

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Dirac's (interaction) picture

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Heisenberg picture

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talk for Šebafest: Hradec Králové, May 10th, 2017

my interest in "pictures" was inspired by Petr Šeba:

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P. Exner and **P. Šeba**, Phys. Lett. A 245 (**1998**) 35

("Probability current tornado loops in three-dimensional scattering"):

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"Writing $\psi(r) = \sqrt{\varrho(r)}\, e^{i\phi(r)}$ **we can . . .** "

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cf. also **P. Šeba** et al, J. Phys. A 32 (**1999**) 8225

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see also *section* H: "The *pilot wave* formulation (de Broglie and Bohm)"

in review "**Nine formulations of quantum mechanics**"

by D. F. Styer et al, Am. J. Phys. 70 (**2002**) 288

and, more recently, by the emergence of new ones:

A. Mostafazadeh, Int. J. Geom. Methods Mod. Phys. 7 (**2010**) 1191
"*Pseudo-Hermitian representation of quantum mechanics*"

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especially *implication* of Theorem 2: "insisting on observability [of $G(t)$] and requiring unitarity [of evolution] **prohibit** [IP]" (= **we cannot . . .**)

aim: a return to “we can”

pictures *in nuce* **(Styer et al)**

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- A. matrix formulation (Heisenberg ’25)
- B. wavefunction formulation (Schrödinger ’26)
- E. density matrix formulation (von Neumann ’27)
- F. second quantization (Dirac ’27, Jordan and Klein ’27, Jordan and Wigner ’28)
- G. variational formulation (Jordan and Klein ’27)
- H. pilot wave formulation (de Broglie ’27, Bohm ’52)
- D. phase-space formulation (Wigner ’32)
- C. path-integral formulation (Feynman ’48)
- I. action/angle formulation (Leacock and Padgett ’83)

♡ Schrödinger picture (SP, maximally economical)

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(a) preparation at $t = t_i = 0$:

$$|\varphi^{(SP)}(0)\succ \in \mathcal{H}^{(T)}$$

(b) of interest:

$$\mathfrak{q}_{(SP)} = \mathfrak{q}^{\dagger}_{(SP)} \neq \mathfrak{q}_{(SP)}(t) \quad \text{and} \quad |\varphi^{(SP)}(t)\succ \in \mathcal{H}^{(T)}$$

(c) evolution law: Schrödinger equation

$$\mathrm{i}\hbar \frac{d}{dt}\,|\varphi^{(SP)}(t)\succ = \mathfrak{h}_{(SP)}\,|\varphi^{(SP)}(t)\succ\ , \qquad \mathfrak{h}_{(SP)} = \mathfrak{h}^{\dagger}_{(SP)}\,,$$

(d) measurement at $t = t_f = T > 0$

$$\prec\varphi^{(SP)}(T)|\mathfrak{q}_{(SP)}|\varphi^{(SP)}(T)\succ$$

$\diamondsuit$ Heisenberg picture (HP, maximally intuitive)

.

origin: **guesswork called "quantization"**

classical $q(t) \quad \longrightarrow \quad$ quantum $Q_{(HP)}(t)$

char. feature: ***constant*** $|\varphi^{(HP)}\rangle \neq |\varphi^{(HP)}(t)\rangle$

.

$$\boxed{|\varphi^{(SP)}(t)\succ = \mathcal{U}_{(HP)}(t)|\varphi^{(HP)}\rangle}$$

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Heisenberg equations to be solved

predictions: $\langle\varphi^{(HP)}|Q_{(HP)}(T)|\varphi^{(HP)}\rangle$

♣ remark: the mind-boggling omission of the Dirac's interaction picture

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IP known from all textbooks but *not put* in the list

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why? speculations:
(1) Haag's theorem
(2) overcomplicated
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$\Longrightarrow$ let's fill the gap (and let's add fresh news)

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♠ **key message: besides the missing tenth IP item**

∃ several other interesting formulations of quantum theory

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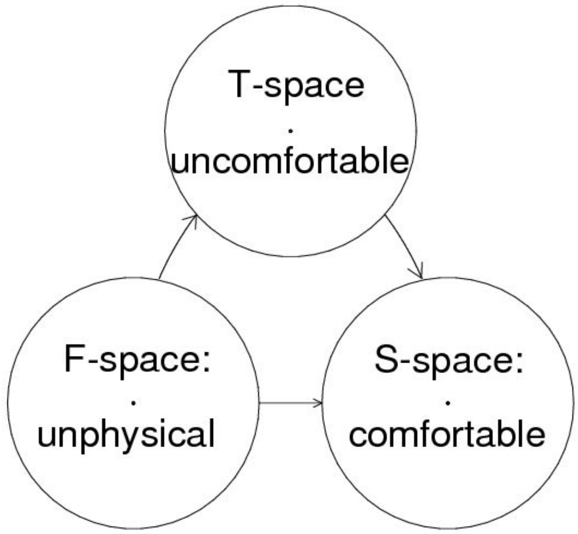
unifying feature: three Hilbert spaces used (PTO)

option I: static Dyson maps Ω



the oldest, Dyson-Schrödinger picture (DSP, 1956)

char. feature: *non-unitary* map $\Omega_{(Dyson)}$ in the

stationary ansatz: $|\varphi^{(SP)}(t)\succ = \Omega_{(Dyson)}|\varphi^{(DSP)}(t)\rangle$

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$\Longrightarrow$ Dyson-Schrödinger equation with $H_{(DSP)} \neq H^{\dagger}_{(DSP)}$ in $\mathcal{H}^{(F)}$

$$\mathrm{i}\hbar\frac{d}{dt}\,|\varphi^{(DSP)}(t)\rangle = H_{(DSP)}\,|\varphi^{(DSP)}(t)\rangle$$

(a) states living in a friendlier space (e.g., non-fermionic, bosonic space in IBM):

$$|\varphi^{(DSP)}(t)\rangle \in \mathcal{H}^{(F)}$$

.

(b) $\mathcal{H}^{(F)}$ = auxiliary, $\boxed{\text{unphysical}}$; observables = non-Hermitian:

$$Q^\dagger_{(DSP)} \neq Q_{(DSP)} = \Omega^{-1}\mathfrak{q}_{(SP)}\Omega \neq Q_{(DSP)}(t)$$

.

(c) ("metric") $\boxed{\Theta = \Omega^\dagger\Omega}$ defines the ultimate, "standard" physical Hilbert space $\mathcal{H}^{(S)}$

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(d) $\Longrightarrow$ return to Hermiticity and to the correct predictions (in $\mathcal{H}^{(S)}$):

.

$$\boxed{\langle\varphi^{(DSP)}(T)|\,\Theta_{(DSP)}\,Q_{(DSP)}|\varphi^{(DSP)}(T)\rangle}$$

the inverted, Buslaev-Schrödinger picture (BSP, 1993)

char. feature: Dyson's flowchart upside down:

false $\mathcal{H}^{(F)}$ and non-Hermitian $H_{(DSP)} \neq H^\dagger_{(DSP)}$ given

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auxiliary map $\Omega_{(Dyson)}$ and textbook Hamiltonian $\mathfrak{h}_{(SP)}$ reconstructed

exactly solvable $\mathcal{PT}-$symmetric model offered

(V. Buslaev and V. Grecchi, J. Phys. A 26 (1993) 5541)

$$H_{(BSP)} = \frac{1}{2}\left(-\frac{d^2}{dx^2} + \frac{j^2-1}{4s^2(x)} + s^2(x)\right) - g^2 s^4(x)\,, \qquad s(x) = x - \mathrm{i}\epsilon$$

B+G got, surprisingly, the closed form of

$$\mathfrak{h}_{(SP)} = -\frac{d^2}{dx^2} + x^2(gx-1)^2 - j(gx-1/2)$$

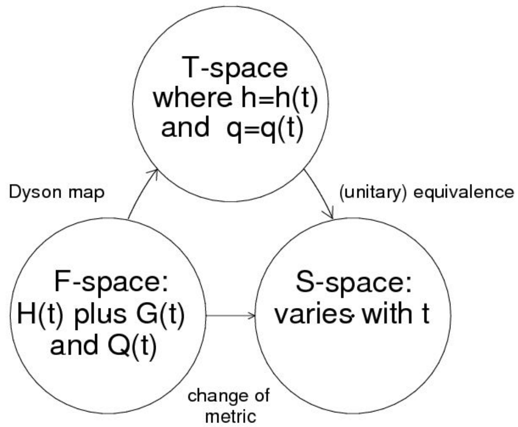
with Ω = change of variables plus Fourier transform.

1998: Bender's $\mathcal{PT}-$symmetric $V(x) = x^2(\mathrm{i}x)^\delta$

(C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243)

option II: non-static Dyson maps $\Omega(t)$

non-static metrics $\Theta(t)$ (IP)



state: two IP vectors in $\mathcal{H}^{(F)}$

(Hermitian) SP states → (non-Hermitian) IP states

$$|\varphi(t)\succ = \Omega(t)|\psi(t)\rangle = \left[\Omega^\dagger(t)\right]^{-1}|\widetilde{\psi}(t)\rangle \in \mathcal{H}^{(P)}\,, \qquad |\psi(t)\rangle, |\widetilde{\psi}(t)\rangle \in \mathcal{H}^{(F)}\,.$$

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(Hermitian) SP Hamiltonian → (non-Hermitian) IP Hamiltonian (real energy)

$$\mathfrak{h}_{(SP)}(t) = \Omega(t)\,H(t)\,\Omega^{-1}(t)$$

evolution law: two Schrödinger equations with $G(t) = H(t) - \Sigma(t)$ and $\Sigma(t) = \mathrm{i}\Omega^{-1}(t)\,[\partial_t\Omega(t)]$

$$\mathrm{i}\frac{d}{dt}\,|\psi(t)\rangle = G(t)\,|\psi(t)\rangle\,, \qquad |\psi(t)\rangle \in \mathcal{H}^{(F)}\,,$$

$$\mathrm{i}\frac{d}{dt}\,|\widetilde{\psi}(t)\rangle = G^\dagger(t)\,|\widetilde{\psi}(t)\rangle\,, \qquad |\widetilde{\psi}(t)\rangle \in \mathcal{H}^{(F)}\,.$$

evolution law for observables

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SP: $\mathfrak{a}(t) = \mathfrak{a}^\dagger(t)$
F-space: $A(t) = \Omega^{-1}(t)\mathfrak{a}(t)\Omega(t)$

$$\mathrm{i}\frac{d}{dt}\,A(t) = A(t)\Sigma(t) - \Sigma(t)A(t) + \Omega^{-1}(t)[\mathrm{i}\dot{\mathfrak{a}}(t)]\Omega(t)\,.$$

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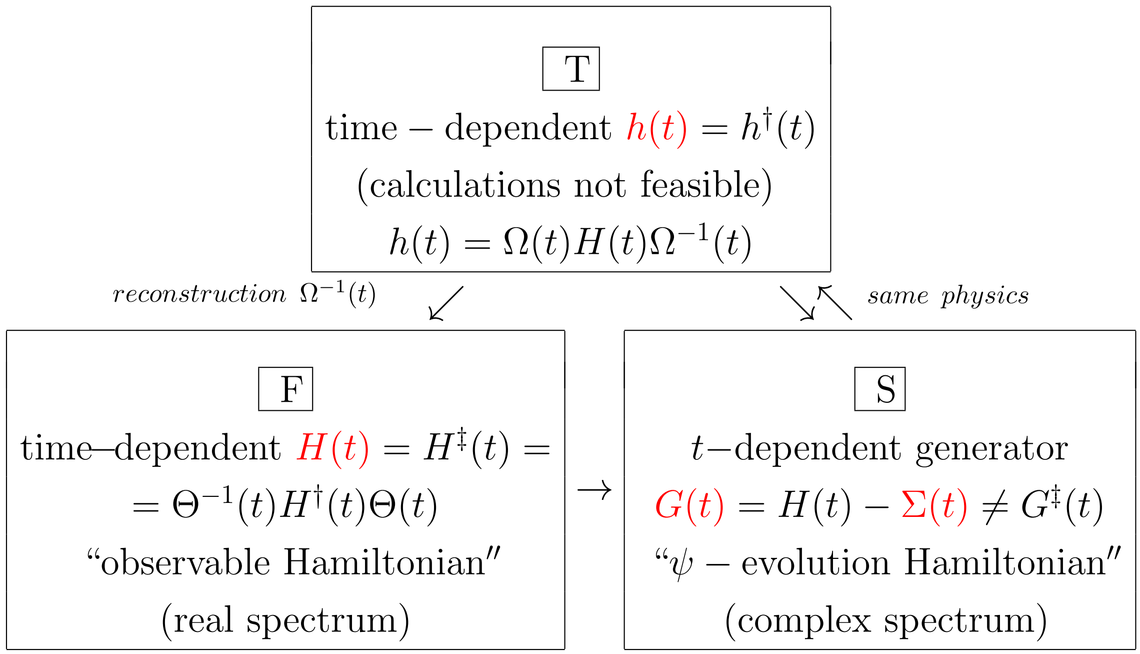
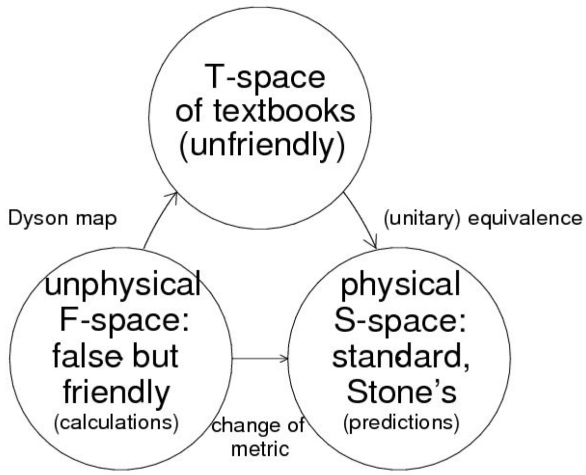
$= \boxed{\text{non-Hermitian Heisenberg equation}}$


T
time − dependent $h(t) = h^\dagger(t)$
(calculations not feasible)
$h(t) = \Omega(t) H(t) \Omega^{-1}(t)$
reconstruction $\Omega^{-1}(t)$
same physics
F
time–dependent $H(t) = H^\ddagger(t) =$
$= \Theta^{-1}(t) H^\dagger(t) \Theta(t)$
"observable Hamiltonian"
(real spectrum)
S
t−dependent generator
$G(t) = H(t) - \Sigma(t) \neq G^\ddagger(t)$
"ψ − evolution Hamiltonian"
(complex spectrum)

III: non-static Dyson maps $\Omega(t)$

static metrics $\Theta(t)$



HP = special case of IP such that $G(t) = 0$

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task: keep wave functions time-independent

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easy to achieve: set $\Sigma(t) = H(t)$

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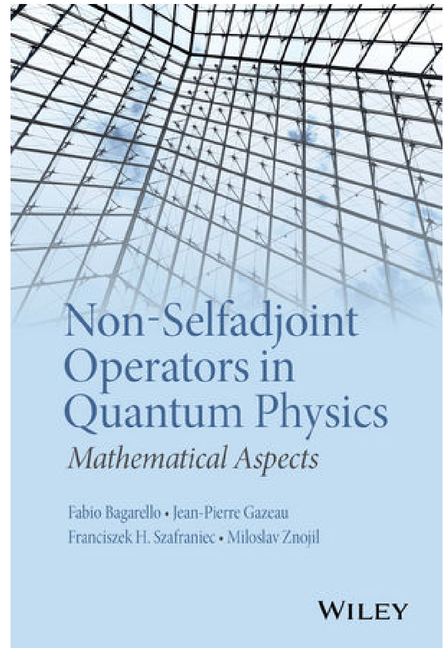
Non-Hermitian Heisenberg representation.

. **MZ, Phys. Lett. A 379 (2015) 2013-2017**

. **(arXiv:1505.01036)**

thanks for your attention;

you may read more about the subject, in chapter "Ideas, people, and trends" by MZ, pp. 7 - 58 of



Petře, happy birthday!